Robust optimal sizing of an hybrid energy stand-alone system

Alain Billionnet(1), Marie-Christine Costa(2), Pierre-Louis Poirion(3)*

(1) ENSIIE-CEDRIC, 1 Square de la Résistance F-91025 Évry Cedex France

(2) ENSTA ParisTech (and CEDRIC-CNAM) 828, Boulevard des Maréchaux 91762 Palaiseau Cedex

(3) CEDRIC-ENSIIE (and ENSTA ParisTech) 1 Square de la Résistance F-91025 Évry Cedex

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Abstract

Todo

1 Introduction

The fast development of renewable energies brought new complex problems of combinatorial optimization in particular as regards the autonomous hybrid energy systems. These systems involve several energy sources as wind, biomass or sun and are not connected to the grid. They are particularly useful in islands as La Réunion [?] and in remote areas as there are in Canada [?]. Mathematical techniques have been used to optimize either the operation of the system [?, ?, ?] or the design of the park which this paper is dealing with. The most recent references on this topic include [?], [?], [?],[?] and [?]. In [?], a methodology is introduced to perform the optimal sizing of an autonomous photovoltaic/wind system. In [?] ??et j ai viré Lambert ?. In [?], integer linear programming is used to optimize the design of wind farm collection networks. In [?] the aim is to determine the types, numbers and placement of wind turbines to install considering investment costs and power production criteria; a mixed-integer nonlinear model is proposed and tested. In [?], an integrated photovoltaic/wind system with battery storage is considered: a heuristic approach is proposed to find the sizes of the wind farm, photovoltaic array and battery.

*Corresponding author: E-mail address: pierre-louis.poirion@ensta-paristech.fr

In this paper, we study a stand-alone hybrid system composed of wind turbines, solar photovoltaic (or PV) panels and batteries. To compensate for a lack of energy from these sources, an auxiliary fuel generator guarantees to meet the demand in every case but its use induces important costs. We consider here that the type of wind turbines, PV panels, and batteries have been predetermined by the users according to the lands on which the park will be settled. The aim is to determine the optimal number of photovoltaic panels, wind turbines and elements of battery to install in order to serve a given demand while minimizing the total cost of investment and use.

Moreover, the stochastic behavior of the solar and wind energy production on the one hand, and the demand on the other hand, needs to search for a robust solution, i.e. a solution which is good enough whatever the scenario that occurs. We assume that there is no known probability distributions of the data and following the approach proposed in [**?**] and [**?**], we consider that the uncertain data can vary between given bounds and that there are limits to the total variation of each kind of data. We propose a mixed integer program in two stage to model the problem: investment decision variables also called here-and-now variables must be fixed in the first stage while operating recourse variables also called wait-and-see variables will be determined once the uncertainty has been revealed. Then we follow the approach proposed in [**?**] to solve the problem.

In Section 2 we describe the system operation and the notations. In Section 3 we give a mixed-integer program for the problem without uncertainty and we prove that a generalisation of this problem is NP-hard. In Section 4 we propose a model in two stages based on mixed-integer programming where the decision variables are integer and the recourse variables are real. In Section 5 we show that, in this case the recourse problem, i.e. the second stage problem, can be solved in polynomial time by using dynamic programming, contrary to the general case. In Section 6 we propose an exact approach based on constraints generation to solve the robust problem. In Section 7 we test our method on real instances obtained in [**?**]. In the last section, we explain how our model can be generalized to select the best ones among different types of wind turbines or PV panel before concluding.

2 System operation

The study concerns period spanning many years but, to be efficient, the optimization model focuses on one year which is decomposed in T time periods of one hour, where a time period t goes from time $t-1$ to time t. Considering that the main part of the data depends on the climate, this allows to take into account the variations of weather during the year. As explained before, the types of wind turbines and PV pannels that will be installed are predetermined. They are defined by their expected nominal output power, respectively E_t^w and E_t^p (in Kw.h) for each time period t, which are functions of the characteristics of the equipments, the land where they will be installed and the mean meteorological data over the past few years for each time period. The costs of a wind turbine and a PV pannel are denoted by C^w and C^p respectively: these costs include purchase and installation costs (reduced to one year according to the lifetime of equipments) and annual maintenance cost including the lease of land.

The purchase and maintenance of the diesel generator induce a fixed cost which is not involved in the optimization. However, its use induces a cost proportional to the energy it provides; it is denoted by C^g (for 1 Kw.h).

The system is described in Figure 1. When weather conditions are favorable, the energy produced by wind turbines and PV panels is sufficient to serve the demand and the excess energy can be used to charge the battery. In case of unfavorable weather, the energy stored in the battery is used to serve the demand. It is only when battery is empty that the fuel generator is used: the cost-in-use of the generator is very high but it allows to meet the demand in any case.

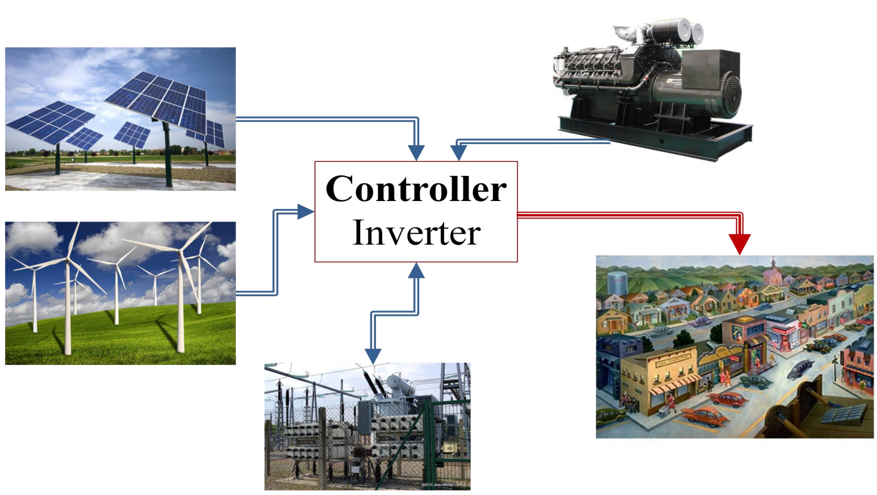

Figure 1: A stand-alone energy park

The battery bank is composed of elements connected together. The cost of one element is denoted by C^b and it has a minimal load denoted by K_{min} and a (maximal) capacity denoted by K_{max}: w.l.o.g., in the following we consider that, for each element, the minimal load is equal to 0 and the capacity is equal to $K = K_{max} - K_{min}$. We assume that the initial load of each element is equal to 0. Each element has a maximum charge per hour, denoted by E^{in}, and a maximum discharge per hour, denoted by E^{out}, in Kw.h. The battery bank operating induces a loss of energy. The rate of return is denoted by $\gamma < 1$: for 1 Kw.h charged in the battery, only γ Kw.h can be effectively used. The physical constraints impose that either all the elements are charging or all are discharging; notice that, since the rate of return is less than 1, it is easy to prove that this battery bank operating is optimal. Then, if there are x^b elements in the battery bank, we can consider that there is only one battery whith capacity equal to $x^b K$, with a maximum charge per hour equal to $x^b E^{in}$ and a maximum discharge per hour equal to $x^b E^{out}$. As in [**?**], we assume that the state of the battery does not change during a time period: either the battery is charging or it is discharging.

From the size of the lands, the characteristics of equipments and their placement on the land, the maximum number N^w_{max} of wind turbines, N^p_{max} of PV panels and N^b_{max} of elements in the battery bank which can be installed have been calculated. The mean expected demand in energy has also been evaluated for each time period and is denoted by D_t, $t = 1, ..., T$.

3 The problem without uncertainty

3.1 the model

If there is no uncertainty, the objective is to determine the design of the park, that is the number of wind turbines, PV panels and elements in the battery bank, in order to meet the demand with a minimal global cost. We propose a mixed-integer model where variables x^w, x^p and x^b are respectively the (integral) number of wind turbines, PV panels and elements in the battery to install in the park, and for $t = 1, ..., T$, variables e^g_t is the amount of energy produced by the generator during the time period t (from $t - 1$ to t), e^{in}_t (resp. e^{out}_t) denote the amount of energy being charged (resp. discharged) in the battery during the time period t and for $t = 0, ..., T$, e^b_t denote the load of the battery at time t.

The problem can be written as the following mixed-integer linear program:

$$
(LP) \left| \begin{array}{ll}
\min_{x,e} C^w x^w + C^p x^p + C^b x^b + C^g \sum_{t=1}^{T} e_t^g & \\
E_t^p x^p + E_t^w x^w - e_t^{in} + \gamma e_t^{out} + e_t^g \geq D_t,\ t = 1, ..., T & (1) \\
e_t^{in} \leq x^b E^{in},\ t = 1, ..., T & (2) \\
e_t^{out} \leq x^b E^{out},\ t = 1, ..., T & (3) \\
e_t^b \leq x^b K,\ t = 1, ..., T & (4) \\
e_t^b = e_{t-1}^b + e_t^{in} - e_t^{out},\ t = 1, ..., T & (5) \\
x^p \leq N_{max}^p & (6) \\
x^w \leq N_{max}^w & (7) \\
x^b \leq N_{max}^b & (8) \\
x^w,\ x^p,\ x^b \ \in \mathbb{N}, & (9) \\
e_t^{in},\ e_t^{out},\ e_t^b,\ e_t^g \in \mathbb{R}^+,\ t = 1...T;\ e_0^b = 0 & (10)
\end{array} \right.
$$

In the objective function, $C^w x^w + C^p x^p + C^b x^b$ corresponds to the investment cost as defined in Section 2, and $C^g \sum_{t=1}^{T} e_t^g$ represents the usage cost. For each time period, constraints (1) impose that the demand is met: the amount of energy obtained from wind and sun is equal to $E_t^w x^w + E_t^p x^p$; $-\gamma e_t^{in} + \gamma e_t^{out} + e_t^g$ is the energy supplied by the battery (see Remark 1) and the auxiliary generator. Constraints (2), (3) and (4) limit the charge, discharge and load of the battery. Constraints (5) gives the load of the battery at time t: it is equal to the load at time $t-1$ (e_{t-1}^b), plus the energy charged in the battery (e_t^{in}) minus the energy provided by the battery (e_t^{out}); as explained in Section 2, we suppose w.l.o.g. that $e_0^b = 0$. Constraints (6), (7) and (8) bounds the number of installed equipments. Finally, constraints (9) and (10) impose that all variables are positive, variables x are integer and variables e are real.

Remark 1. *There is an optimal solution of* (LP) *such that* $e_t^{in} e_t^{out} = 0$.

Proof. Assume that $e_t^{in} > 0$ and $e_t^{out} > 0$ in an optimal solution S and consider $\hat{S}$ obtained form S by modifying only two variables: $\hat{e}_t^{in} = e_t^{in} - e_t^{out}$ and $\hat{e}_t^{out} = 0$ if $e_t^{in} \geq e_t^{out}$, and $\hat{e}_t^{out} = e_t^{out} - e_t^{in}$ and $\hat{e}_t^{in} = 0$ if $e_t^{in} < e_t^{out}$. Constraints (2), (3) and (5) are verified for S and stay verified for $\hat{S}$ since both variables are decreased while $e_t^{in} - e_t^{out}$ remains constant. In addition, in constraints (1), $-\hat{e}_t^{in} + \gamma \hat{e}_t^{out} = -e_t^{in} + e_t^{out}$ if $e_t^{in} \geq e_t^{out}$ and $-\hat{e}_t^{in} + \gamma \hat{e}_t^{out} = -\gamma e_t^{in} + \gamma e_t^{out}$ if $e_t^{in} < e_t^{out}$. In the two cases, $-\hat{e}_t^{in} + \gamma \hat{e}_t^{out} \geq -e_t^{in} + \gamma e_t^{out}$ and constraints (1) are verified for $\hat{S}$. The value of the objective function and all the other variables are unchanged and $\hat{S}$ is an optimal solution of (LP).

□

From Remark 1 from any an optimal solution of (LP) we can deduce another optimal solution such that $e_t^{in} e_t^{out} = 0$ for all $t = 1, ..., T$. That is why these constraints, implying that the battery is either in charge or in discharge during a time period, may be omitted in (LP).

In the following, we denote by $\mathcal{P}^x$ the set defined by constraints (6,7,8 9) (on x) and by $\mathcal{P}^e$ the set defined by constraints (2, 3, 4, 5, 10) (on e).

Here, the problem has only three integer variables and it can be easily solved with a mixed-integer linear programming software, but we show in Section 3.2 that the problem is NP-hard when there are n sources of energy.

3.2 Complexity of the problem

Let us consider a more general model where there are n sources of renewable energy. We will call (LP_{gen}) the resulting model. In this case, one unit of source i costs C^i and produces a quantity E_t^i of energy during period t, $t = 1, ..., T$. The objective function becomes $\sum_{i=1}^n C^i x^i + C^b x^b + C^g \sum_{t=1}^T e_t^g$. Constraint (1) becomes $\sum_{i=1}^n E_t^i x^i - e_t^{in} + \gamma e_t^{out} + e_t^g \geq D_t, t = 1...T$ and constraints (6)-(7) are replaced by $x^i \leq N_{max}^i, i = 1, ..., n$. Constraints (2)-(5), (8) and (10) are retained.

We are going to show that this generalized problem is NP-hard. Thus, there is no polynomial time algorithm solving (LP_{gen}) unless $P = NP$.

Proposition 1. (LP_{gen}) *belongs to the class of NP-hard problems.*

Proof. We will show that the bounded knapsack problem (BKP) reduces to (LP_{gen}). Consider the decision problem ($DBKP$) associated with (BKP):

$$(DBKP) \begin{cases} \textit{Data: } n, a_1, a_2, ..., a_{n+1}, c_1, c_2, ..c_{n+1}, u_1, u_2, ..., u_{n+1}, b, V \text{ in } \mathbb{N} \\ \textit{Question: } \text{Is there a vector } (y_1, ..., y_{n+1}) \in \mathbb{N}^{n+1} \text{ such that} \\ \sum_{i=1}^{n+1} a^i y^i \geq V,\ \sum_{i=1}^{n+1} c^i y^i \leq b \text{ and } y_i \leq u_i,\ i = 1, ..., n+1. \end{cases}$$

Now we define the decision problem (DLP_{gen}) associated with (LP_{gen}):

$$(DLP_{gen}) \begin{cases} \textit{Data: } n, T, C, C^b, C^g, E^{in}, E^{out}, e_0^b, \gamma, K, N_{max}^b, D_t, C^i, E_t^i, N_{max}^i, \\ t = 1, ..., T, i = 1, ..., n. \\ \textit{Question: } \text{Is there a vector } (x^1, ..., x^n) \in \mathbb{N}^n, \text{ an integer } x^b, \text{ and} \\ \text{nonnegative reals } e_t^{in}, e_t^{out}, e_t^b, e_t^g, t = 1, ..., T, \text{ satisfying the} \\ \text{constraints of } (LP_{gen}) \text{ and such that the value of the objective} \\ \text{function is less than or equal to } C? \end{cases}$$

From an instance of $(DBKP)$, let us construct the following instance of (DLP_{gen}): $T = 1$; there are n sources of energy with $C^i = c_i$, $E_1^i = a_i$ and $N_{max}^i = u_i$ for $i = 1, ..., n$; $C^b = c_{n+1}$, $K = a_{n+1}$, $N_{max}^b = u_{n+1}$, $E^{in} = E^{out} = e_0^b = a_{n+1}$,$\gamma = 1$; $C^g = 2b$; $D_1 = V$;$C = b$.

Notice that we can set $e_1^{in} = 0$ and $e_1^{out} = a_{n+1}$, which results in $e_1^b = 0$ from constraint (5). Then constraints (2), (3), (4) and (5) are verified and the corresponding problem is to determine if there are $(x^1, ..., x^n) \in \mathbb{N}^n$, $x^b \in \mathbb{N}$, e_1^{in}, e_1^{out}, e_1^b, $e_1^g \in \mathbb{R}^+$ such that the following set of constraints is satisfied :

$$\left| \begin{array}{l} \sum_{i=1}^{n} c_i x^i + c_{n+1} x^b + 2b e_1^g \leq b \quad (**) \\ \sum_{i=1}^{n} a_i x^i + a_{n+1} x^b + e_1^g \geq V \quad (*) \\ x^i \leq u_i, \ i = 1, ..., n \\ x^b \leq u_{n+1} \\ x^b, x^i (i = 1, ..., n) \in \mathbb{N}, e_1^g \in \mathbb{R}^+ \ . \end{array} \right.$$

Let $y_1, y_2, ..., y_{n+1}$ be a solution of $(DBKP)$. It is clear that $x^i = y_i$ for $i = 1, ..., n$, $x^b = y_{n+1}$ and $e_1^g = 0$ is a solution of (DLP_{gen}).

Conversely, let $((x^i)_{1 \leq i \leq n}, x^b, e_1^g)$ be a solution of (DLP_{gen}), i.e. of the above set of constraints. According to constraint (*), $\sum_{i=1}^n a_i x^i + a_{n+1} x^b \geq V - e_1^g$ and, according to constraint (**), $e_1^g \leq 1/2$; then we have $\sum_{i=1}^n a_i x^i + a_{n+1} x^b \geq V - 1/2$. Since the quantities $\sum_{i=1}^n a_i x^i + a_{n+1} x^b$ and V are integers, we finally obtain: $\sum_{i=1}^n a_i x^i + a_{n+1} x^b \geq V$. Thus, the vector y defined by $y_i = x^i$ for $i = 1, ..., n$, $y_{n+1} = x^b$ is a solution of $(DBKP)$, and the two problems are equivalent.

Since $(DBKP)$ is NP-complete [**?**], we have proved that the decision problem (DLP_{gen}) associated with the generalized problem (LP_{gen}) is also NP-complete.

□

4 A robust model

Now, let us consider that a part of the data are uncertain. For sake of clarity, we assume for the moment that there is uncertainty only on the demand which can vary in a given domain $\mathfrak{D}$. Our robustness objective is to find a feasible solution (x, e) that minimizes the total cost involved by the worst possible scenario of $\mathfrak{D}$ in connection with x. We can state the robust problem as the following mathematical program:

$$(RP) \left| \begin{array}{l} \min_{x \in \mathcal{P}^x} \; C^p x^p + C^w x^w + C^b x^b \\ \qquad + \max_{d \in \mathfrak{D}} \min_{e \in \mathcal{P}^e} C^g \sum_{t=1}^{T} e_t^g \\ \qquad\qquad E_t^p x^p + E_t^w x^w - e_t^{in} + \gamma e_t^{out} + e_t^g \geq d_t \;\; t = 1, ..., T \end{array} \right.$$

Variables x are the decision variables and variables e are the recourse variables. For any feasible x, the following program $R(x)$ is called the "Recourse Program":

$$R(x) \left| \begin{array}{l} \max_{d \in \mathfrak{D}} \; \min_{e \in \mathcal{P}^e} C^g \sum_{t=1}^{T} e_t^g \\ \qquad E_t^p x^p + E_t^w x^w - e_t^{in} + \gamma e_t^{out} + e_t^g \geq d_t \;\; t = 1, ..., T \end{array} \right.$$

As in [**?**] or [**?**], we define the uncertainty set by:

$$\mathfrak{D} = \{d \in \mathbb{R}_+^T : \; d_t = D_t + \delta_t \Delta_t, \; \sum_{t=1}^{T} \delta_t \leq \bar{\delta}, \; 0 \leq \delta_t \leq 1, \; \forall t = 1, ..., T\},$$

where D_t, Δ_t and $\bar{\delta}$ are data, $\bar{\delta}$ being integer. Then, the demand d_t will vary between D_t and $D_t + \Delta_t$, for $t = 1, ..., T$. Δ_t is the maximum variation of d_t, δ_t represents the uncertainty on d_t, and $\bar{\delta}$ fixes a bound to the cumulated variations. Since we only consider the worst scenarios, δ_t vary between 0 and 1 (and not between -1 and 1 as it could be expected). If $\bar{\delta} \geq T$, then whatever the values of x, the worst-case scenario will be $d_t = D_t + \Delta_t$ for all t; if $\bar{\delta} = 0$ then there is only one scenario: $d_t = D_t$ for all t; thus $\bar{\delta}$ will be chosen between 0 and T: the choice of $\bar{\delta}$ is discussed in Section 7. This definition of the uncertainty implies that the total variation of the demand relative to its reference value D is bounded.

We can generalize the model to the case where there is also uncertainty on the generation of solar and wind energies. Let e_t^p (resp. e_t^w) be the uncertain energy produced by PV arrays (resp. wind turbines). Similarly to $\mathfrak{D}$, we define the uncertainty sets $\mathfrak{E}^p$ and $\mathfrak{E}^w$ associated to E^p and E^w as:

$$\mathfrak{E}^p = \{e^p : \; e_t^p = E_t^p - \phi_t \Phi_t, \; \sum_{t=1}^{T} \phi_t \leq \bar{\phi}, \; 0 \leq \phi_t \leq 1, \; \forall t = 1, ..., T\}, and$$

$$\mathfrak{E}^w = \{e^w : \; e_t^w = E_t^w - \omega_t \Omega_t, \; \sum_{t=1}^{T} \omega_t \leq \bar{\omega}, \; 0 \leq \omega_t \leq 1, \; \forall t = 1, ..., T\}.$$

As for the demand, since we only consider the worst scenarios, for given uncertainty budgets $\bar{\phi}$ and $\bar{\omega}$, we only consider cases where $e_t^p \leq E_t^p$ and $e_t^w \leq E_t^w$. From [**?**] the method proposed hereafter to solve the problem for an uncertain demand (right-hand side of contraints) can be extended for uncertain data on the left-hand side of the constraints. For sake of clarity, we present the method only for uncertain demands, but the tests in Section 7 are computed for uncertainty on e^p and e^w as well.

For fixed values of x, the recourse problem $R(x)$ is generally a difficult problem [**?**]; nevertheless, we show in the next section that it can be solved in polynomial time in our case.

5 Solving the recourse problem

From [**?**], we know that there is an optimal solution of $R(x)$ such that $\delta_t \in \{0, 1\}$ for all $t = 1, ..., T$ and we can rewrite $R(x)$ as follows:

$$R(x) \left| \begin{array}{l} \max\limits_{\substack{\sum_{t=\tau}^{T} \delta_t \leq \zeta \\ \delta_t \in \{0,1\},\ t=\tau,...,T}} \quad \min\limits_{e \in \mathcal{P}^e} C^g \sum_{t=1}^{T} e_t^g \\ \qquad\qquad\qquad E_t^p x^p + E_t^w x^w - e_t^{in} + \gamma e_t^{out} + e_t^g \geq D_t + \delta_t \Delta_t \ \ t = 1, ..., T \end{array} \right.$$

Clearly, for any x, the worst scenario is obtained with $\sum_{t=1}^{T} \delta_t = \bar{\delta}$ and then it will be determined by setting $\delta_t = 1$ for $\bar{\delta}$ periods and $\delta_t = 0$ for the others, or equivalently by setting $d_t = D_t + \Delta_t$ for $\bar{\delta}$ periods and $d_t = D_t$ for the others. The problem now is to select the $\bar{\delta}$ periods for which $\delta_t = 1$. We propose a polynomial dynamic programming approach to answer this question and solve the recourse problem.

We have to solve the recourse problem for given values of x, thus $E_t^p x^p + E_t^w x^w$ is fixed and we denote $\hat{D}_t = D_t - E_t^p x^p - E_t^w x^w$, $\hat{d}_t = d_t - E_t^p x^p - E_t^w x^w = \hat{D}_t + \delta_t \Delta_t$. If $\hat{d}_t \geq 0$ then sun and wind energies are sufficient to serve the demand, otherwise the battery and/or generator must be used. In addition, since x_b is fixed, we assumed w.l.o.g. that $x^b = 1$, or equivalently we set $K \leftarrow x_b K$, $E^{in} \leftarrow x_b E^{in}$ and $E^{out} \leftarrow x_b E^{out}$. Since the unit diesel cost does not depend on t, we can determine the battery load at time t (e_t^b) and the amount of energy supplied by the generator during period t (e_t^g), from the battery load at $t-1$ (e_{t-1}^b) and the amount of energy either in excess or in lack ($\hat{d}_t$). We have the following proposition :

Proposition 2. *For given values of* $\hat{d}$ *and* $\beta = e^b_{t-1}$ *(*$0 \leq \beta \leq K$*), for* $t = 1, ..., T$*, we have:*

$$\left.\begin{aligned} e^b_t &= f(\hat{d}_t, \beta) = \max\left(\beta - \frac{\hat{d}_t}{\gamma}, \beta - E^{out}, 0\right) \\ e^g_t &= g(\hat{d}_t, \beta) = \hat{d}_t - \gamma \min\left(\beta, E^{out}, \frac{\hat{d}_t}{\gamma}\right) \end{aligned}\right\} \quad \textit{if } \hat{d} \geq 0$$

$$\left.\begin{aligned} e^b_t &= f(\hat{d}_t, \beta) = \min\left(\beta - \hat{d}_t, \beta + E^{in}, K\right) \\ e^g_t &= g(\hat{d}_t, \beta) = 0 \end{aligned}\right\} \quad \textit{if } \hat{d} < 0$$

Proof. During time period t,

- If $\hat{d}_t \geq 0$, then the energy produced by wind turbines and PV panels is not sufficient to serve the demand. The battery discharges a quantity of energy equal to $\min\left(\beta, E^{out}, \frac{\hat{d}_t}{\gamma}\right)$, and the generator must provide an amount of energy equal to $g(\hat{d}_t, \beta) = \hat{d}_t - \gamma \min\left(\beta, E^{out}, \frac{\hat{d}_t}{\gamma}\right) \geq 0$.
- If $\hat{d}_t < 0$, then the energy produced is greater than the demand and the system stores in the battery an amount of energy equal to $\min\left(-\hat{d}_t, K - \beta, E^{in}\right)$. The generator is not used and $g(\hat{d}_t, \beta) = \hat{d}_t - \gamma \min\left(\beta, E^{out}, \frac{\hat{d}_t}{\gamma}\right) = 0$.

The battery load at time t, $f(\hat{d}_t, \beta)$, is immediately deduced from the amount of energy charged or discharged from the battery during time period t. □

The algorithm operates from $\tau = T$ to $\tau = 1$ and considers at each step a "truncated recourse problem" $R_x(\tau, \zeta, \beta)$ defined on the $(T - \tau + 1)$ last time periods. At each step, it looks for an optimal operating for the considered period, i.e. from time $\tau - 1$ to time T, in function of the two parameters β and ζ: β is the battery load at time $\tau - 1$, i.e. at the beginning of period τ, and ζ is the uncertainty budget for these periods, that is the number of time periods with $\delta_t = 1$ among the $(T - \tau + 1)$ time periods; then, $\bar{\delta}$ represents the global "uncertainty budget" and the only value of ζ to consider for $\tau = 1$ is $\zeta = \bar{\delta}$.

Then $R_x(\tau, \zeta, \beta)$ can be written as the following mathematical program where $\hat{d}_t = \hat{D}_t + \delta_t \Delta_t$:

$$R_x(\tau,\zeta,\beta) \left| \begin{array}{ll} \max\limits_{\substack{\sum_{t=\tau}^{T}\delta_t\le\zeta \\ \delta_t\in\{0,1\},\ t=\tau,...,T}} \min\limits_{e} C^g \sum_{t=\tau}^{T} e_t^g & \\ -e_t^{in} + \gamma e_t^{out} + e_t^g \ge \hat{d}_t,\ t=\tau,...,T & (1) \\ e_t^{in} \le E^{in},\ t=\tau,...,T & (2) \\ e_t^{out} \le E^{out},\ t=\tau,...,T & (3) \\ e_t^b \le K,\ t=\tau,...,T & (4) \\ e_t^b = e_{t-1}^b - e_t^{out} + e_t^{in},\ t=\tau,...,T & (5) \\ e_t^{in},\ e_t^{out},\ e_t^b,\ e_t^g \in \mathbb{R}_+,\ t=\tau,...,T & (10) \end{array} \right.$$

Notice that (τ, ζ, β) represent the "current state" of the decision process and that $R(x) = R_x(1, \bar{\delta}, 0)$. Let us denote $v(R_x(\tau, \zeta, \beta))$ by $v(\tau, \zeta, \beta)$. We obtain the following recurrence relation verified by $v(\tau, \zeta, \beta)$ for $0 \le \beta \le K$:

$v(\tau, 0, \beta) = C^g g(\hat{D}_\tau, \beta) + v(\tau+1, 0, f(\hat{D}_\tau, \beta))$, for $1 \le \tau < T$,

$v(T, \zeta, \beta) = C^g g(\hat{D}_T + \zeta\Delta_T, \beta)$, for $0 \le \zeta \le 1$,

$v(\tau, \zeta, \beta) = \max(C^g g(\hat{D}_\tau, \beta) + v(\tau+1, \zeta, f(\hat{D}_\tau, \beta)), C^g g(\hat{D}_\tau + \Delta_\tau, \beta) +$
$v(\tau+1, \zeta-1, f(\hat{D}_\tau + \Delta_\tau, \beta)))$, for $1 \le \tau < T$ and $0 < \zeta \le T - \tau + 1$.

Now, let us study $v(\tau, \zeta, \beta)$. We are going to prove that there is a constant B $(0 \le B \le K)$ such that $v(\tau, \zeta, \beta)$ is a function of β linearly decreasing on $[0, B]$ and constant on $[B, K]$ (see Figure 2).

Proposition 3. *For any $\tau,\ 1 \le \tau \le T$, any $\zeta, 0 \le \zeta \le T - \tau + 1$, and any $\beta \in [0, K]$, there exist $B \ge 0$ and $C \ge 0$ such that*

$$v(\tau, \zeta, \beta) = C - \gamma C^g \min(\beta, B),$$

with $B = \frac{v(\tau,\zeta,0) - v(\tau,\zeta,K)}{\gamma C^g}$ and $C = v(\tau, \zeta, 0)$.

Proof. The complete proof is given in Annex. We just give here the main ideas it uses.

First, the proof is given for $\zeta = 0$: in this case, the problem reduces to a problem without uncertainty since $\delta_t = 0$ for all $t = \tau$ to T. The proof is by induction on t, from T to $T - \tau + 1$; first we prove the existence of B and C and then we compute them. When $\zeta = T - \tau + 1$ we have $\delta_t = 1$ for all $t = \tau$ to T and the problem reduces to a problem without uncertainty too. The proof is similar.

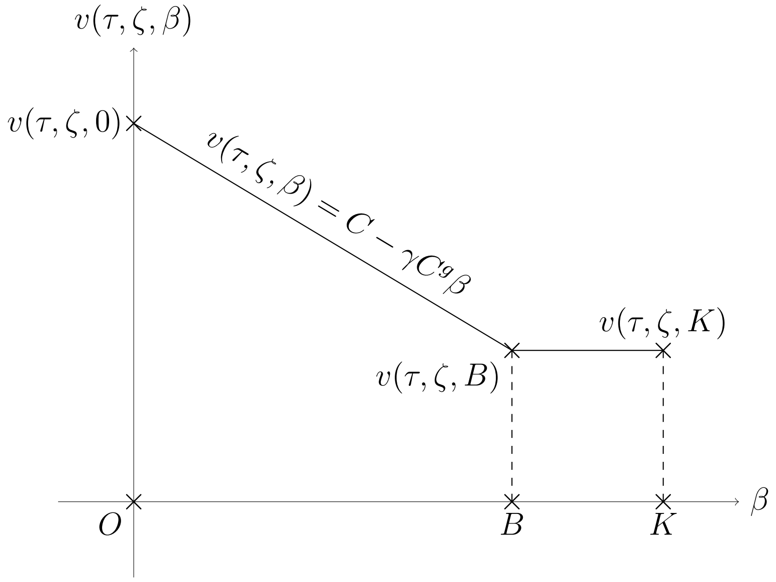

Figure 2: Variations of $v(\tau, \zeta, \beta)$ in function of β

For values of ζ between 0 and $T - \tau + 1$, the proof is again by induction and it uses the results obtained for $\zeta = 0$ and $\zeta = T - \tau + 1$ before computing B and C. □

Proposition 3 gives us an easy way to compute $v(\tau, \zeta, \beta)$. Indeed, for any β, the function $v(\tau, \zeta, \beta)$ is entirely determined by the two values $v(\tau, \zeta, 0)$ and $v(\tau, \zeta, K)$. We can therefore use two nested dynamic programming procedures computing $v(\tau, \zeta, 0)$ and $v(\tau, \zeta, K)$ for each (τ, ζ) and following the pattern shown on Figure 3.

Notice that since the computation of the initial states is $O(T^2)$, the complexity of the algorithm is $O(\bar{\delta}(T - \bar{\delta} + 1) + T^2)$. Since $\bar{\delta} \leq T$, this algorithm is polynomial.

6 Solving the robust problem

The robust problem can be rewritten as:

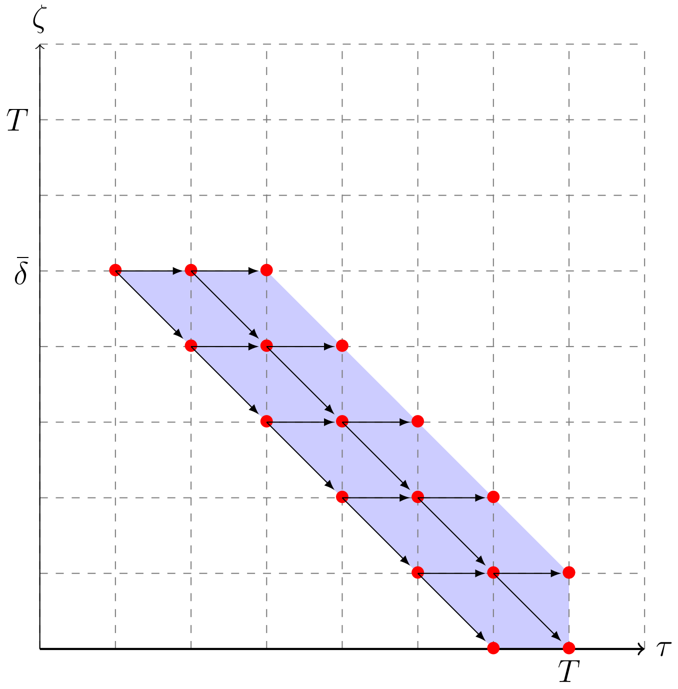

Figure 3: Pattern of calculation of $v(\tau, \zeta, \beta)$ for fixed β.

$$(RP)\left|\begin{array}{l}\min\limits_{x} C^p x^p + C^w x^w + C^b x^b + v(R(x)) \\ \qquad x^p \leq N^p_{max} \\ \qquad x^w \leq N^w_{max} \\ \qquad x^b \leq N^b_{max} \\ \qquad x^b,\ x^w,\ x^p \in \mathbb{N}\end{array}\right.$$

where $v(R(x))$ is the optimal value of $R(x)$ that can be obtained in polynomial time. Thus, we could use a branch and bound algorithm based on x^p, x^w and x^b to solve (RP). Nevertheless, we preferred following the approach proposed in [**?**] which proved to be very efficient to solve real instances. For any $x \in \mathcal{P}^x$ and $d \in \mathfrak{D}$, we define the following linear program:

$$R(x,d)\left|\begin{array}{l}\min\limits_{e \in \mathcal{P}^e} C^g \sum\limits_{t=1}^{T} e^g_t \\ \qquad -e^{in}_t + \gamma e^{out}_t + e^g_t \geq d_t - E^w_t x^w - E^p_t x^p,\ t = 1..T\ (1)\end{array}\right.$$

Let us associate the dual variables λ to constraints (1) and α, β, μ, π respectively to constraints (2), (3), (5) and (4) of $\mathcal{P}^e$. Following the approach proposed in [?] or [?], we dualize $R(x,d)$. Then $R(x)$ becomes a $\max\max$ instead of a $\max\min$ problem and we can reformulate $R(x)$ to obtain a quadratic maximization program $DR(x)$ such that $v(DR(x)) = v(R(x))$:

$$DR(x) \left| \begin{array}{lr} \displaystyle\max_{\alpha,\beta,\delta,\lambda,\mu,\pi} \sum_{t=1}^{T}[(D_t - \hat{D}_t - \delta_t\Delta_t)\lambda_t \quad -x^b E^{in}\alpha_t - x^b E^{out}\beta_t - x^b K\pi_t] & \\ \text{s.c. } \lambda_t \leq C^g, & t = 1,...,T \ (13) \\ \quad -\lambda_t - \alpha_t + \mu_t \leq 0, & t = 1,...,T \ (14) \\ \quad \gamma\lambda_t - \beta_t - \mu_t \leq 0, & t = 1,...,T \ (15) \\ \quad \mu_{t+1} - \mu_t - \pi_t \leq 0, & t = 1,...,T \ (16) \\ \quad \displaystyle\sum_{t=1}^{T} \delta_t \leq \bar{\delta} & (18) \\ \quad \delta_t \in \{0,1\}, & t = 1,...,T(19) \\ \quad \alpha_t, \beta_t, \lambda_t, \mu_t, \pi_t \geq 0 & t = 1,...,T(17) \end{array} \right.$$

Finally, we linearize the quadratic terms in the objective function by substituting the variable ν_t to the product $\delta_t\lambda_t$. Finally, we obtain the following mixed-integer linear program

$$LDR(x) \left| \begin{array}{lr} \displaystyle\max_{\alpha,\beta,\delta,\lambda,\mu,\nu,\pi} \sum_{t=1}^{T}[(D_t - \hat{d}_t)\lambda_t - x^b \quad E^{in}\alpha_t - x^b E^{out}\beta_t - x^b K\pi_t + \Delta_t\nu_t] & \\ \text{s.c. } \lambda_t \leq C^g, & t = 1,...,T(13) \\ \quad -\lambda_t - \alpha_t + \mu_t \leq 0, & t = 1,...,T(14) \\ \quad \gamma\lambda_t - \beta_t - \mu_t \leq 0, & t = 1,...,T(15) \\ \quad \mu_{t+1} - \mu_t - \pi_t \leq 0, & t = 1,...,T(16) \\ \quad \displaystyle\sum_{t=1}^{T} \delta_t \leq \bar{\delta} & (18) \\ \quad \delta_t \in \{0,1\}, & t = 1,...,T(19) \\ \quad \nu_t \leq C^g\delta_t, & t = 1,...,T(20) \\ \quad \nu_t \leq \lambda_t, & t = 1,...,T(21) \\ \quad \alpha, \beta, \lambda, \mu, \nu, \pi \geq 0 & (17) \end{array} \right.$$

Notice that the linear program above can be solved in polynomial time. Indeed, the optimal values of $\delta_t,\ t = 1,...,T$, have be obtained following the dynamic programming approach of Section 5. Then the optimal solution of $LDR(x)$ can be determined in polynomial time, by solving the (continuous) linear program obtained by fixing in $DR(x)$ the variables δ_t to their optimal values. Let $\mathcal{P}_Q$ be the polyhedron defined by the constraints $(13),...,(21)$ of $LDR(x)$ where we replace (19) by $0 \le \delta_t \le 1$, and let $(\mathcal{P}_Q)_I = conv(\mathcal{P}_Q \cap \{\delta \in \{0,1\}^T\})$, be the convex hull of the feasible solution of $LDR(x)$. Notice that this convex hull does not depend on x. $(\mathcal{P}_Q)_I$ being a polyhedron we can rewrite the robust problem (RP) as the following linear program:

$$
PROB \left| \begin{array}{l} \min\limits_{x,z} C^p x^p + C^w x^w + C^b x^b + z \\ s.c.\ z \ge \sum\limits_{t=1}^{T} [(D_t - E_t^s x^s - E_t^w x^w)\lambda_t^s - x^b E^{in} \alpha_t^s \\ \qquad\qquad - x^b E^{out} \beta_t^s - x^b K \pi_t^s + \Delta_t \nu_t^s], \ \ s = 1...S. \\ \qquad x^b \le N_{max}^b, \\ \qquad x^p \le N_{max}^p, \\ \qquad x^w \le N_{max}^w, \\ \qquad z \ge 0,\ x^b,\ x^w,\ x^p \in \mathbb{N} \end{array} \right.
$$

where $S = |\mathcal{S}|$ and $\mathcal{S} = \{(\alpha^s, \beta^s, \lambda^s, \mu^s, \nu^s, \pi^s)_{1 \le s \le S}\}$ is the set of extreme points of $(\mathcal{P}_Q)_I$. However, due to the potentially tremendous number of constraints, we solve $(PROB)$ by a constraint generation algorithm as in [**?**] or [**?**]. Initially, we consider a subset $\mathcal{S}_0$ of $\mathcal{S}$; at a step k, we consider a subset $\mathcal{S}^k$ of $\mathcal{S}$ and we solve a relaxed program $(PROB)^k$ of $(PROB)$, called *master problem*, which consists in solving $(PROB)$ with the subset of constraints corresponding to $\mathcal{S}^k$. The obtained solution in denoted by (x^k, z^k).
Then we solve $LDR(x^k)$, called *slave problem*, to check if (x^k, z^k) is optimal. If not, then a new constraint is added, i.e. an extreme point is added to $\mathcal{S}^k$ (See Algorithm 1).
On the basis that the number of extreme points of $(\mathcal{P}_Q)_I$ is finite, one can prove that this algorithm converges in a finite number of steps.

7 Results and conclusion

The proposed wind-PV model was applied for providing power to an off-grid network in Montana, USA, for several real instances given in [**?**].

The test have been performed on a Bi-pro. Intel Nehalem XEON 5570 at 2.93 GHz with 24 Go of RAM.

Algorithm 1 Constraint generation algorithm

1: $(\alpha^0, \beta^0, \lambda^0, \mu^0, \nu^0, \pi^0) = 0$. Set $L \leftarrow -\infty, U \leftarrow +\infty, k \leftarrow 1$.

2: Solve the master problem :

$$(PROB)^k \left| \begin{array}{l} \min_{x,z} C^p x^p + C^w x^w + C^b x^b + z \\ s.c.\ z \geq \sum_{t=1}^{T} [(D_t - E_t^s x^s - E_t^w x^w)\lambda_t^s - x^b E^{in} \alpha_t^s \\ \quad - x^b E^{out} \beta_t^s - x^b K \pi_t^s + \Delta_t \nu_t^s],\ \ 0 \leq s \leq k-1. \\ \quad x^b \leq N_{max}^b, \\ \quad x^p \leq N_{max}^p, \\ \quad x^w \leq N_{max}^w, \\ \quad z \geq 0,\ x^b,\ x^w,\ x^p \in \mathbb{N}. \end{array} \right.$$

Let (x^k, z^k) be the obtained solution.
$L \leftarrow \alpha x^k + z^k$.

3: Solve $LDR(x^k)$. Let $(\lambda^k, \delta^k, \nu^k)$ be the optimal solution.

$$U \leftarrow \min\{U, \alpha x^k + v(DR(x^k))\}.$$

if $U = L$, **then** return (x^k, z^k) **else** go to 4.

4: Add the constraint

$$z \geq \sum_{t=1}^{T} (\bar{d}_t - (Ax)_t)\lambda_t^k + \Delta_t \nu_t^k,$$

to the master problem $(PR)^k$, $k \leftarrow k+1$ and go to 2.

The components system are wind turbines of type BWC XL.1 with a rated power of 1.24 kW, PV panels of size 1000 kW and batteries Trojan L16P. Table 1 gives the main parameter values. The number T of time periods is equal to 8760 which is the number of hours in one year. The unit costs, C^w, C^p and C^b are computed in the following way: $\frac{\text{purchase cost}}{\text{lifetime}}$+annual O&R cost(Operations and Regulatory). The level of uncertainty for the demand during time perdiod t, Δ_t is equal to ten percent of the mean demand at time period t, and the bounds on the number of PV panels, wind turbines and batteries are all equal to 500. The other values are directly extracted from [**?**]. We do not recall here the 8760 hourly mean values of the demand and the wind and solar energies.

We consider two cases; in the first one there is uncertainty only on the demand, in the second one the uncertainty also concerns wind and solar energy. In both cases, the recourse problem is solved either by using the dynamic programming algorithm of Section 5 or Cplex. Assume first that uncertainty concerns only the demand. Seven values of $\bar{\delta}$ are tested, results are given in Table 2:

for each value, we give the optimal number of wind turbines, PV panels and elements in the battery. We also give the optimal value of the robust problem. As expected, the optimal value of the robust problem increases as a function of $\bar{\delta}$,

C^w	C^p	C^b	C^g	K	γ	E^{in}	E^{out}	$N^p_{max}, N^w_{max}, N^b_{max}$
295.0$	280.0$	26.0$	3.9$	2.16	0.85	0.11	2.16	500

Table 1: Main data values

	$\bar{\delta}$ $(T = 8760)$						
	0	100	500	700	850	1000	8760
x^w	45	46	48	48	48	48	48
x^p	64	65	67	67	67	67	67
x^b	467	470	499	499	499	499	499
Cost (in $)	49874.5	51564	54540.3	54901.9	54901.9	54901.9	54901.9

Table 2: Park design and total associated robust cost as a function of the global level of uncertainty $\bar{\delta}$

until a certain threshold (here 700), threshold which is an increasing function of the demand d, and then remains constant. Indeed, the optimal value of the robust problem cannot decrease when the global uncertainty level increases since any solution for $\bar{\delta} = \hat{\delta}$ is admissible for $\bar{\delta} > \hat{\delta}$. Furthermore, once the park is sufficient to cover the demand for some critical sequel of periods, it is also sufficient to cover any other periods. Therefore the cost remains constant as soon as $\bar{\delta}$ is large enough to cover the critical periods. In Table 3, we compare the CPU times required by the algorithm when the recourse problem is solved either by using the polynomial time dynamic algorithm of Section 5 or by using CPLEX. We notice that for intermediate values of $\bar{\delta}$ the dynamic programming approach is

	$\bar{\delta}$						
	0	100	500	700	850	1000	8760
CPU(s) dynamic programming	82	90	97	95	94	102	67
CPU(s) Cplex	46	88	7422	375	204	65	22

Table 3: Comparison of the two approches from the CPU time point of view

much faster than the approach using Cplex. In particular, for values of $\bar{\delta}$ between 0 and 850, the Cplex approach can take several hours to solve the problem, while the dynamic programming approach only needs a few minutes. Furthermore, with Cplex, the CPU time increases until $\bar{\delta} = 900$ and then decreases until $\bar{\delta} = 8760$.

Indeed, when $\bar{\delta} > 850$, less and less nodes are explored in the branch and bound.

Now we consider that there are also uncertainty in the values of wind and solar energy. In Table 4, we give the results and the number of iterations of the constraint generation algorithm, for several values of $(\bar{\delta}, \bar{\phi}, \bar{\omega})$.

	iterations	x^w	x^p	x^b	Cost
$\bar{\delta} = 0, \bar{\phi} = 500, \bar{\omega} = 0$	27	49	69	476	51844.3
$\bar{\delta} = 0, \bar{\phi} = 0, \bar{\omega} = 500$	27	48	66	465	51348.4
$\bar{\delta} = 8760, \bar{\phi} = 0, \bar{\omega} = 0$	26	49	72	499	54901.9
$\bar{\delta} = 0, \bar{\phi} = 8760, \bar{\omega} = 0$	30	48	71	456	51852.0
$\bar{\delta} = 0, \bar{\phi} = 0, \bar{\omega} = 8760$	29	51	64	469	51438.2
$\bar{\delta} = 0, \bar{\phi} = 8760, \bar{\omega} = 8760$	26	50	71	471	53362.7
$\bar{\delta} = 100, \bar{\phi} = 100, \bar{\omega} = 100$	29	47	67	468	53743
$\bar{\delta} = 500, \bar{\phi} = 500, \bar{\omega} = 500$	28	49	74	500	58182.6

Table 4: Park design and total associated robust cost with energy and demand uncertainties

Here all the results have been obtained by using Cplex, since if we take into account the 3 parameters, $\bar{\delta}, \bar{\phi}$ and $\bar{\omega}$, the dynamic programming approach becomes intractable for large value of T, although the algorithm remains polynomial.
As previously, the optimal value of the robust problem increases as a function of $\bar{\delta}, \bar{\phi}$ and $\bar{\omega}$, until a certain threshold. Furthermore, we notice that the combined influence of uncertainties in the energy generated by wind turbines and PV-panels induces a larger cost augmentation than the sum of the ones induced by considering separate uncertainties in wind and solar energy generation. Indeed, once the $\bar{\delta}$ critical periods for the demand uncertainty is set, the new uncertainty resulting from $\phi > 0$ or $\omega > 0$ will increase the cost induced by the previous $\bar{\delta}$ critical periods.

Annex : proof of Proposition 3

First, we give the proof for $\zeta = 0$:

Proposition 4. *If $\zeta = 0$ then for for any τ, $1 \leq \tau \leq T$ and any $\beta \in [0, K]$, there exist $B \geq 0$ and $C \geq 0$ such that*

$$v(\tau, \zeta, \beta) = C - \gamma C^g \min(\beta, B),$$

with $B = \frac{v(\tau,\zeta,0)-v(\tau,\zeta,K)}{\gamma C^g}$ *and* $C = v(\tau,\zeta,0)$.

Proof. The problem reduces to a problem without uncertainty: if $\zeta = 0$ then $\delta_t = 0$ for all $t = \tau, ..., T$. We first prove that for each $\tau \in \{1, ..., T\}$ there are two constants B and C such that:

$$v(\tau, 0, \beta) = C - \gamma C^g \min(\beta, B), \tag{1}$$

then we verify the expression of B and C. From the recurrence relation (1), we have, for $\tau < T$:

$$v(\tau, 0, \beta) = C^g g(\hat{D}_\tau, \beta) + v(\tau + 1, 0, f(\hat{D}_\tau, \beta)).$$

We then proceed by induction from T to 1.

- Step 1: $\tau = T$, $v(T, 0, \beta) = C^g g(\hat{D}_T, \beta)$.
  If $\hat{D}_T < 0$, then $v(T, 0, \beta) = 0$ ($B_1 = 0, C_1 = 0$).
  Else $v(T, 0, \beta) = C^g \hat{D}_T - \gamma C^g \min\left(\beta, E^{out}, \frac{\hat{D}_T}{\gamma}\right)$ ($B_1' = \min\left(E^{out}, \frac{\hat{D}_T}{\gamma}\right)$, $C_1' = C^g \hat{D}_T$.)
- Step 2: $\tau = \bar{\tau} + 1$.
  Assume there are $B_2 \geq 0$ and $C_2 \geq 0$ such that

  $$v(\bar{\tau} + 1, 0, \beta) = C_2 - \gamma C^g \min(\beta, B_2).$$

- Step 3: $\tau = \bar{\tau}$,

  $$v(\bar{\tau}, 0, \beta) = C^g g(\hat{D}_{\bar{\tau}}, \beta) + C_2 - \gamma C^g \min(f(\hat{D}_{\bar{\tau}}, \beta), B_2).$$

  - If $\hat{D}_{\bar{\tau}} \geq 0$, we have:

  $$\begin{aligned} v(\bar{\tau}, 0, \beta) &= C^g \hat{D}_{\bar{\tau}} - \gamma C^g \min\left(\beta, E^{out}, \frac{\hat{D}_{\bar{\tau}}}{\gamma}\right) \\ &+ C_2 - \gamma C^g \min\left(\max\left(\beta - \frac{\hat{D}_{\bar{\tau}}}{\gamma}, \beta - E^{out}, 0\right), B_2\right) \\ &= (C^g \hat{D}_{\bar{\tau}} + C_2) \\ &- \gamma C^g \left(\min\left(\beta, E^{out}, \frac{\hat{D}_{\bar{\tau}}}{\gamma}\right) + \min\left(\max\left(\beta - \frac{\hat{D}_{\bar{\tau}}}{\gamma}, \beta - E^{out}, 0\right), B_2\right)\right) \end{aligned}$$

By considering the three cases $\beta \leq \min\left(E^{out}, \frac{\hat{D}_{\bar{\tau}}}{\gamma}\right)$,
$\min\left(E^{out}, \frac{\hat{D}_{\bar{\tau}}}{\gamma}\right) < \beta \leq B_2 + \min\left(E^{out}, \frac{\hat{D}_{\bar{\tau}}}{\gamma}\right)$, and $B_2 + \min\left(E^{out}, \frac{\hat{D}_{\bar{\tau}}}{\gamma}\right) < \beta$, it is easy to verify that:

$$\begin{aligned} v(\bar{\tau}, 0, \beta) &= (C^g \hat{D}_{\bar{\tau}} + C_2) - \gamma C^g \min\left(\beta, B_2 + \min\left(E^{out}, \frac{\hat{D}_{\bar{\tau}}}{\gamma}\right)\right) \\ &= C_3 - \gamma C^g \min(\beta, B_3) \end{aligned}$$

with $B_3 = \min\left(B_2 + \min\left(E^{out}, \frac{\hat{D}_{\bar{\tau}}}{\gamma}\right), K\right)$ (we can choose $B_3 \leq K$ since $\beta \leq K$) and $C_3 = (C^g \hat{D}_{\bar{\tau}} + C_2)$.

– If $\hat{D}_{\bar{\tau}} < 0$, then:

$$\begin{aligned} v(\bar{\tau}, 0, \beta) &= 0 + C_2 - \gamma C^g \min\left(\min(\beta - \hat{D}_{\bar{\tau}}, \beta + E^{in}, K), B_2\right) \\ &= C_2 - \gamma C^g \min(\beta - \hat{D}_{\bar{\tau}}, \beta + E^{in}, K, B_2) \\ &= C_2 - \gamma C^g \min(\beta + \min(-\hat{D}_{\bar{\tau}}, E^{in}), \min(K, B_2)) \\ &= (C_2 - \gamma C^g \min(-\hat{D}_{\bar{\tau}}, E^{in})) - \gamma C^g(\beta, \min(K, B_2) - \min(-\hat{D}_{\bar{\tau}}, E^{in})) \\ &= C'_3 - \gamma C^g \min(\beta, B'_3) \end{aligned}$$

$$\text{with} \begin{cases} B'_3 = \min(K, B_2) - \min(-\hat{D}_{\bar{\tau}}, E^{in}) \text{ and } C'_3 = C_2 - \gamma C^g \min(-\hat{D}_{\bar{\tau}}, E^{in}) \\ \text{ if } \min(K, B_2) \geq \min(-\hat{D}_{\bar{\tau}}, E^{in}) \\ B'_3 = 0 \text{ and } C'_3 = C_2 - \gamma C^g \min(-\hat{D}_{\bar{\tau}}, E^{in}) - \gamma C^g(\min(K, B_2) - \min(-\hat{D}_{\bar{\tau}}, E \\ \text{ if } \min(K, B_2) < \min(-\hat{D}_{\bar{\tau}}, E^{in}) \end{cases}$$

Now, let us verify the expression of B and C. Taking $\beta = 0$ in (2) gives $C = v(\bar{\tau}, 0, 0)$. Furthermore, $v(\bar{\tau}, 0, B) = v(\bar{\tau}, 0, 0) - \gamma C^g B = v(\bar{\tau}, 0, K)$, and we have

$$B = \frac{v(\bar{\tau}, 0, 0) - v(\bar{\tau}, 0, K)}{\gamma C^g}.$$

The proof if $\zeta = T - \tau + 1$ is similar and is not given here. □

The proof for $\zeta = T - \tau + 1$ is very similar and is not given here. Just notice that $\delta_t = 1$ for all t. We now prove the proposition for any value of (ζ).

Proposition 5. *VOIR NUMEROTATION For any τ, $1 \leq \tau \leq T$, any $\zeta, 0 \leq \zeta \leq T - \tau + 1$, and any $\beta \in [0, K]$, there exist $B \geq 0$ and $C \geq 0$ such that*

$$v_{(}\tau, \zeta, \beta) = C - \gamma C^g \min(\beta, B),$$

with $B = \frac{v(\tau,\zeta,0) - v(\tau,\zeta,K)}{\gamma C^g}$ and $C = v(\tau, \zeta, 0)$.

Proof. We prove the proposition by recurrence on (τ, ζ).

- Step 1: from Proposition 3, The proposition is true for any τ if $\zeta = 0$ or $\zeta = T - \tau + 1$ (i.e. for $(\tau, \zeta) \in \{(T, 0), (T-1, 0), ..., (1, 0), (T, 1), (T-1, 2), ..., (T-i, i+1), ..., (T-\bar{\delta}-1, \bar{\delta})\}$)

- Step 2: assume it is true for $(\tau+1, \bar{\zeta}-1)$ and for $(\tau+1, \bar{\zeta})$, we have

$$\forall \beta \in [0, K], v(\tau+1, \bar{\zeta}-1, \beta) = C_2 - \gamma C^g \min(\beta, B_2),$$
$$v(\tau+1, \bar{\zeta}, \beta) = C_2' - \gamma C^g \min(\beta, B_2'),$$

- Step 3: from the recurrence relation (1) we have:

$$v(\tau, \bar{\zeta}, \beta) = \max(C^g g(\hat{D}_\tau, \beta) + v(\tau+1, \bar{\zeta}, f(\hat{D}_\tau, \beta)),$$
$$C^g g(\hat{D}_\tau + \Delta_\tau, \beta) + v(\tau+1, \bar{\zeta}-1, f(\hat{D}_\tau + \Delta_\tau, \beta))).$$

Let $Q = C^g g(\hat{D}_\tau, \beta) + v(\tau+1, \bar{\zeta}, f(\hat{D}_\tau, \beta))$.

  - If $\hat{D}_\tau \geq 0$:

$$\begin{aligned} Q &= C^g \hat{D}_\tau - \gamma C^g \min\left(\beta, E^{out}, \frac{\hat{D}_\tau}{\gamma}\right) \\ &+ C_2' - \gamma C^g \min\left(\max\left(\beta - \frac{\hat{D}_\tau}{\gamma}, \beta - E^{out}, 0\right), B_2'\right) \\ &= (C^g \hat{D}_\tau + C_2') \\ &- \gamma C^g \left[\min\left(\beta, E^{out}, \frac{\hat{D}_\tau}{\gamma}\right) + \min\left(\max\left(\beta - \frac{\hat{D}_\tau}{\gamma}, \beta - E^{out}, 0\right), B_2'\right)\right] \end{aligned}$$

    As in the proof of Proposition 3, we get

$$\begin{aligned} Q &= (C^g \hat{D}_\tau + C_2') - \gamma C^g \min\left(\beta, B_2' + \min\left(E^{out}, \frac{\hat{D}_\tau}{\gamma}\right)\right) \\ &= C_3' - \gamma C^g \min(\beta, B_3') \end{aligned}$$

    with $B_3' = \min\left(B_2' + \min\left(E^{out}, \frac{\hat{D}_\tau}{\gamma}\right), K\right)$ and $C_3' = (C^g \hat{D}_\tau + C_2')$.

– If $\hat{D}_\tau < 0$, then:

$$\begin{aligned}
Q &= 0 + C_2' - \gamma C^g \min\Big(\min(\beta - \hat{D}_\tau, \beta + E^{in}, K), B_2'\Big) \\
&= C_2' - \gamma C^g \min(\beta - \hat{D}_\tau, \beta + E^{in}, K, B_2') \\
&= C_2 - \gamma C^g \min(\beta + \min(-\hat{D}_\tau, E^{in}), \min(K, B_2)) \\
&= C_3' - \gamma C^g \min(\beta, B_3')
\end{aligned}$$

$$\text{with} \begin{cases} B_3' = \min(K, B_2') - \min(-\hat{D}_\tau, E^{in}) \text{ and } C_3' = C_2' - \gamma C^g \min(-\hat{D}_\tau, E^{in}) \\ \text{ if } \min(K, B_2') \geq \min(-\hat{D}_\tau, E^{in}) \\ B_3' = 0 \text{ and } C_3' = C_2' - \gamma C^g \min(-\hat{D}_\tau, E^{in}) - \gamma C^g (\min(K, B_2') - \min(-\hat{D}_\tau, E \\ \text{ if } \min(K, B_2') < \min(-\hat{D}_\tau, E^{in}) \end{cases}$$

Let $R = C^g g(\hat{D}_\tau + \Delta_\tau, \beta) + v(\tau + 1, \bar{\zeta} - 1, f(\hat{D}_\tau + \Delta_\tau, \beta))$. We prove similarly that there exists $C_3" \geq 0$, and $B_3" \geq 0$, such that

$$R = C_3" - \gamma C^g \min(\beta, B_3").$$

Therefore,

$$v(\tau, \bar{\zeta}, \beta) = \max(C_3' - \gamma C^g \min(\beta, B_3'), C_3" - \gamma C^g \min(\beta, B_3")).$$

By considering the four cases: $C_3' < C"_3$ and $B_3' < B"_3$; $C_3' < C"_3$ and $B_3' \geq B"_3$; $C_3' \geq C"_3$ and $B_3' < B"_3$; $C_3' \geq C"_3$ and $B_3' \geq B"_3$, it is easy to verify that there is $B_3,\ B_3 \in [B_3', B"_3]$, such that:

$$v(\tau, \bar{\zeta}, \beta) = C_3 - \gamma C^g \min(\beta, B_3),$$

with $C_3 = \max(C_3', C"_3)$. we conclude the recurrence referring to the scheme shown on figure 2. Indeed, by proposition 3, the proposition is true for $(\tau, \zeta) \in \{(T, 0), (T, 1)\}$, and it is easy to verify that if the proposition is true for $(\tau, \zeta) \in \{(\bar{\tau}+1, 0), (\bar{\tau}+1, 1), ..., (\bar{\tau}+1, T-(\bar{\tau}+1)+1)\}$, then it is true for $(\tau, \zeta) \in \{(\bar{\tau}, 0), (\bar{\tau}, 1), ..., (\bar{\tau}, T-\bar{\tau})\}$ and for $(\tau, \zeta) = (\bar{\tau}, T-\bar{\tau}+1)$, by proposition 3.

□