Minimal graphs for matching extensions

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Abstract

Given a positive integer n we find a graph $G = (V, E)$ on $|V| = n$ vertices with a minimum number of edges such that for any pair of non adjacent vertices x, y the graph $G - x - y$ contains a (almost) perfect matching M. Intuitively the non edge xy and M form a (almost) perfect matching of G. Similarly we determine a graph $G = (V, E)$ with a minimum number of edges such that for any matching $\bar{M}$ of the complement $\bar{G}$ of G with size $\lfloor \frac{n}{2} \rfloor - 1$, $G - V(\bar{M})$ contains an edge e. Here $\bar{M}$ and the edge e of G form a (almost) perfect matching of $\bar{G}$.

We characterize these minimal graphs for all values of n.

Keywords: maximum matching, matching extension, expandable graph, completable graph.

1 Introduction

We shall consider here a kind of reliability problem which occurs rather naturally in a context where some elements of a complex system may break down either due to attacks or simply to technical failures.We want to protect a subset of elements (as small as possible) in order to keep the system working in spite of possible failures occurring in the rest of the system.

To give a formulation in terms of graphs, we introduce definitions and notations. Given a simple finite graph $G = (V, E)$ with n vertices $v_1, v_2, \ldots, v_n$ and m edges, we denote by $\bar{G} = (V, \bar{E})$ the complement of G. For any subset $F \subseteq E$, $V(F)$ is the set of endpoints of the edges in F. For any subset $X \subseteq V$ the subgraph induced by X is denoted by $G[X]$. We write $G - X = G[V \setminus X]$ and $G - v$ for $G - \{v\}$. The union of two graphs G_1, G_2 on disjoint vertex sets without any edges between them is written $G_1 + G_2$. $N_G(v)$ is the set of neighbors of a vertex v in G; $\delta_G(v) = |N_G(v)|$ is the degree of v in G; a p-vertex is a vertex of degree p in G; if $\delta_G(v) = n - 1$ then v is *universal*. For any nonempty subset $A \subseteq V$ we denote by $N_G(A)$ the set of vertices $v \in V \setminus A$ having a neighbor in A, i.e. $N_G(A) = \bigcup_{v \in A} N_G(v) \setminus A$. Let A,

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B be disjoint sets of vertices. We denote by $m_G(A, B)$ the number of edges linking A and B.

A subset $M \subseteq E$ is a *matching* if no two edges in M are incident to a same vertex; $\mu(G)$ is the maximum cardinality of a matching in G. G has a *perfect* matching if $\mu(G) = n/2$ and an *almost perfect* matching if $\mu(G) = (n-1)/2$.

For all definitions related to graphs, see [4].

We intend to determine for two given positive integers d, n a graph $G = (V, E)$ on n vertices with a minimum number of edges, such that to any matching $\bar{M}$ of d edges of $\bar{G}$ one can associate a matching of $\lfloor n/2 \rfloor - d$ edges in $G - V(\bar{M})$. Hence if the edges of $\bar{M}$ would be edges in G, then $\bar{M} \cup M$ would be a (almost) perfect matching of G. Notice that a feasible set E of edges always exists: take for instance for E the edges of a complete graph on n vertices from which we remove a matching of size d.

In our paper we determine the minimum size of *expandable* graphs G (corresponding to the case $d = 1$); these are graphs such that for any edge xy in $\bar{E}$, the subgraph $G - x - y$ has a (almost-)perfect matching. Similarly we determine the minimum size of *completable* graphs G (corresponding to the case $d = \lfloor n/2 \rfloor - 1$); these are graphs such that for any matching $\bar{M}$ of $\bar{G}$ with $|\bar{M}| = \lfloor n/2 \rfloor - 1$ there exists an edge $uv \in G - V(\bar{M})$.

In our reliability interpretation the edges of these minimal graphs G are the ones which should be protected so that one could extend the matchings $\bar{M}$ of size d to (almost)-perfect matchings in spite of failures in $\bar{G}$.

Various concepts of matching extension have been studied. Some consider these extensions in special classes of graphs [1, 5, 11]. In [10, 11] several properties related to perfect matchings are examined. It is the case of d-extendable graphs defined as graphs in which every matching of size d can be extended to a perfect matching. In particular for $d = 1$, one requires that for any edge xy, $G - x - y$ has a perfect matching [9]. A graph is *bicritical* if for any pair $\{x, y\}$ of vertices, xy being an edge or not, $G - x - y$ has a perfect matching. Notice that the graphs considered there have a perfect matching. Clearly a bicritical graph is 1-extendable and also expandable. A claw $K_{1,3}$ is expandable but not 1-extendable and a cycle C_6 is 1-extendable but not expandable.

It is worth underlining that to our knowledge matching extensions by edges of G or $\bar{G}$ have not been associated with the optimization of the size of the graphs. This is the main motivation for this research.

In the next section we will characterize the expandable graphs of n vertices with a minimum number of edges. The case where the expandable graphs are constrained to be connected is treated in the third section. Then Section 4 will be devoted to completable graphs on n vertices with a minimum number of edges. Finally we will mention in the conclusion some variations and generalizations.

2 Minimal expandable graphs

We want to find a graph G with a minimum number of edges such that for every pair u, v of non adjacent vertices of G it is always possible to extend the non-edge uv to a perfect (or almost perfect) matching using only edges of G that are not incident to u or v, formally $\mu(G - u - v) = \lfloor n/2 \rfloor - 1$.

We say that G is *expandable* if for any non-edge $uv \notin E$ there exists a matching M of $G - u - v$ with $|M| = \lfloor n/2 \rfloor - 1$.

An expandable graph $G = (V, E)$ on n vertices with a minimum number of edges is a *Minimum Expandable Graph.* The size $|E|$ of its edge set is denoted by $Exp(n)$. The set of minimal expandable graphs of order n is called $MEG(n)$.

Since the problem is trivial for $n \leq 3$ we shall assume $n \geq 4$.

Proposition 2.1 *For $4 \leq n \leq 7$ we have:*

- $Exp(4) = 3$ *and* $MEG(4) = \{K_{1,3}, \bar{K}_{1,3}\}$*;*
- $Exp(5) = 3$ *and* $MEG(5) = \{K_3 + 2K_1\}$*;*
- $Exp(6) = 6$ *and* $MEG(6) = \{2K_3\}$*;*
- $Exp(7) = 6$ *and* $MEG(7) = \{2K_3 + K_1, C_5 + K_2\}$.

Proof: Let $n = 4$. One can verify that $K_{1,3}$ and its complement $\bar{K}_{1,3}$ are expandable. Suppose that there exists $G = (V, E) \in MEG(4)$ with $|E| = 2$: then G has two non adjacent 1-vertices v_1, v_2; so $\mu(G - v_1 - v_2) = 0 < 1$. The only graph with three edges non isomorphic to $K_{1,3}$ or $\bar{K}_{1,3}$ is P_4, and P_4 is not expandable.

Let $n = 5$. One can verify that $K_3 + 2K_1$ is expandable. Suppose that there exists $G = (V, E) \in MEG(5)$ with $|E| = 2$: then G has two non adjacent 1-vertices v_1, v_2; so $\mu(G - v_1 - v_2) = 0 < 1$. The only non isomorphic graphs with 3 edges are $K_3 + 2K_1, P_4 + K_1, P_3 + K_2, K_{1,3} + K_1$. Among them only $K_3 + 2K_1$ is expandable.

Let $n = 6$. One can verify that $2K_3$ is expandable. Suppose that there exists $G = (V, E) \in MEG(6)$ with $|E| \leq 5$: if G has a 1-vertex v_1, its neighbor v_2 must be universal otherwise $\mu(G - v_2 - v_i) < 2, v_i \notin N_G(v_2)$. But $G = K_{1,5}$ is clearly not expandable. So G has a 0-vertex and then the five remaining vertices must induce K_5 which has more than six edges.

We prove that the only graph in $MEG(6)$ is $2K_3$. Suppose that there exists $G \in MEG(6)$ and $G \neq 2K_3$. It cannot have a 0-vertex. If G has a 1-vertex then its neighbor must be universal and G consists of a spanning star and an additional edge; such a G is not expandable. It follows that all vertices have degree two and thus $G \in \{C_6, 2K_3\}$ but C_6 is not expandable, hence $G = 2K_3$.

Let $n = 7$. One can verify that $2K_3 + K_1$ and $C_5 + K_2$ are expandable. Suppose that there exists $G = (V, E) \in MEG(7)$ with $|E| \leq 5$: If there exists a 0-vertex u

then $G-u$ must be expandable and from above $|E| \geq 6$. So there are at least four 1-vertices and two of them v_1, v_2 are in two different connected components then $\mu(G-w_1-w_2) < 2$ where w_1, w_2 are the neighbors of v_1, v_2.

We prove that $MEG(7) = \{2K_3 + K_1, C_5 + K_2\}$. Suppose that there exists $G = (V, E) \in MEG(7)$, $|E|= 6$, and $G \neq 2K_3 + K_1, C_5 + K_2$. If G has one 0-vertex u then $G-u$ must be expandable: so $G-u = 2K_3$ and $G = 2K_3 + K_1$. It follows that the number k of 1-vertices in G is at least two.

Two 1-vertices cannot have a common neighbor otherwise G must be a spanning star which is clearly not expandable. Moreover, the neighbors of 1-vertices must induce a clique: if $k > 2$, since $|E| = 6$, then $k = 3$ and there is a 0-vertex: a contradiction.

So G has exactly two 1-vertices and five 2-vertices. Hence $G \in \{C_5 + K_2, P_3 + C_4, P_4 + C_3\}$: $P_4 + C_3$ and $P_3 + C_4$ are not expandable, and $K_2 + C_5$ has been excluded. $\square$

We now consider the case $n \geq 8$ even.

Property 2.1 *If $G = (V, E) \in MEG(n)$ with $n \geq 8$ even and $|E| \leq \frac{3}{2}n - 1$ then G is 2-edge-connected.*

Proof: First we assume that G is not connected. G cannot have an even component C, otherwise $\mu(G-u-v) < n/2-1$ where $u \in C, v \notin C$. G cannot have three (odd) components otherwise by taking any pair of non adjacent vertices u, v in two odd components we get $\mu(G-u-v) < n/2-1$. So G consists of two odd components. Any odd component C is a clique else for any pair of vertices $u, v \in C$ with $uv \notin C$ we have $\mu(G-u-v) < n/2-1$. Let p be the size of one component. We have $|E| = \binom{p}{2} + \binom{n-p}{2}$ which is minimal for $p = n/2$. However, as $n \geq 8$, we have $2\binom{n/2}{2} \geq \frac{3}{2}n$. So G is connected.

Suppose that there exists an edge uv such that $G-uv$ consists of two components C_1, C_2, $u \in C_1, v \in C_2$. Let $n_i = |V(C_i)|, i = 1, 2$. W.l.o.g. $n_2 \geq n_1$. If both n_1, n_2 are even then there exists a pair of vertices u, w with $w \in C_2, w \neq v$, such that $\mu(G-u-w) < n/2-1$ so both n_1, n_2 are odd.

The vertex u, resp. v, is universal in C_1, resp. C_2, else G is not expandable. We have $n_2 \geq 5$ and in C_2 there are four vertices $v_1, v_2, v_3, v_4 \neq v$.

First, suppose that $n_1 \geq 3$, so in C_1 there are two vertices $u_1, u_2 \neq u$. Now in $G-u-v_1$, resp. $G-v-u_1$ there are $(n_1-1)/2$, resp. $(n_2-1)/2$, edges making a perfect matching M_1 in C_1-u, resp. M_2 in C_2-v. W.l.o.g. we can suppose that $v_3v_4 \in M_2$. If $v_1v_3 \notin E$, resp. $v_1v_4 \notin E$, in $G-v_1-v_3$, resp. $G-v_1-v_4$, to get $\mu(G-v_1-v_3) = n/2-1$, resp. $\mu(G-v_1-v_4) = n/2-1$, u and v have to be matched and the vertex v_4, resp. v_3, must be adjacent to a vertex $z \in C_2, z \neq v, v_1, v_3$, resp. $z \in C_2, z \neq v, v_1, v_4$. Hence we have $|E| \geq (n-1)+(n_1-1)/2+(n_2-1)/2+2 = \frac{3}{2}n$.

Second, suppose that $n_1 = 1$. We have $n_2 \geq 7$ and $C_2 = \{v, v_1, v_2, \ldots, v_{n-2}\}$. In $G-u-v_1$ there are $(n_2-3)/2 = (n-4)/2$ edges making a matching M_2 in C_2-v-v_1: w.l.o.g. we can suppose that $v_3v_4, v_5v_6 \in M_2$.

We have $m_G(\{v_i\}, \{v_3, v_4, v_5, v_6\}) \leq 2, i = 1, 2$, else $|E| \geq (n-1)+\frac{n-4}{2}+3 = \frac{3n}{2}$. Let $v_1v_j \notin E$ and $v_2v_k \notin E$ for some $j, k \in \{3, 4, 5, 6\}$. Then $m_G(\{v_2\}, \{v_i, i = 3, ..., n-2\}) \geq 1$ else $\mu(G-v_1-v_j) < \frac{n}{2}-1$ and $m_G(\{v_1\}, \{v_i, i = 3, ..., n-2\}) \geq 1$

else $\mu(G - v_2 - v_k) < \frac{n}{2} - 1$. Now $|E| \geq (n-1) + \frac{n-4}{2} + 2 = \frac{3n}{2} - 1$ and so E cannot contain anymore edges and $m_G(\{v_1\}, \{v_3, v_4, v_5, v_6\}) \leq 1$, $m_G(\{v_2\}, \{v_3, v_4, v_5, v_6\}) \leq 1$. W.l.o.g., $v_1v_3 \notin E, v_2v_3 \notin E, v_2v_6 \notin E$. Since $v_4v_5 \notin E$, the vertex v_3 must be adjacent to a vertex $z \in C_2$, $z \neq v, v_1, v_2, v_4, v_5$ else $\mu(G - v_4 - v_5) < \frac{n}{2} - 1$, and then $|E| \geq \frac{3n}{2}$. □

A direct consequence of this property is that all degrees in G are at least 2.

Property 2.2 *If $G = (V, E) \in MEG(n)$, n even and $|E| \leq \frac{3}{2}n - 1$ then any 2-vertex belongs to a triangle.*

Proof: If $N_G(v) = \{u, w\}$ and $uw \notin E$ then $\mu(G - u - w) < n/2 - 1$; so $uw \in E$. □

Property 2.3 *If $G = (V, E) \in MEG(n)$ with $n \geq 6$ even and $|E| \leq \frac{3}{2}n - 1$ then for all 2-vertices $u, v \in V, u \neq v$, we have $N_G(u) \neq N_G(v)$.*

Proof: Suppose that $N_G(u) = N_G(v) = \{x, y\}$. Necessarily $xy \in E$. If $zx \notin E$ for some $z \in V - \{u, v, y\}$ then $\mu(G - z - x) < n/2 - 1$. By symmetry both x, y are universal and $2|E| = \Sigma_{v \in V} \delta(v) \geq 2(n-1) + 2(n-2) = 2(2n-3) > 2(\frac{3}{2}n - 1)$. □

For the proof of the next proposition, we need the following definitions.

We define the family of hammocks $\mathcal{H}(n)$: for any even integer $n \geq 4$, the hammock $\mathcal{H}(n)$ is the graph with n vertices and edge set $E = \{v_iv_{i+1}, 1 \leq i \leq n-1\} \cup \{v_nv_1\} \cup \{v_iv_{n+2-i}, 1 < i \leq n/2\}$. See Figure 1.

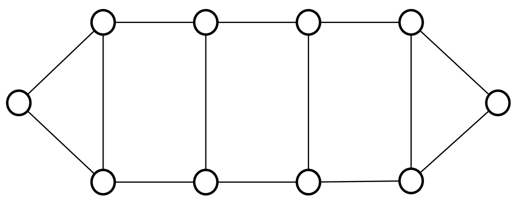

Figure 1: $\mathcal{H}(10)$ the hammock on 10 vertices.

We define the family of fish $\mathcal{F}(n)$: for any even integer $n \geq 6$, the fish $\mathcal{F}(n)$ is the graph with n vertices where $\{v_i : 1 \leq i \leq n-2\}$ induces the hammock $\mathcal{H}(n-2)$ and $\{v_1, v_{n-1}, v_n\}$ induces K_3. See Figure 2.

We define the family of candies $\mathcal{C}(n)$: for any even integer $n \geq 8$, the candy $\mathcal{C}(n)$ is the graph with n vertices where $\{v_i : 1 \leq i \leq n-2\}$ induces the fish $\mathcal{F}(n-2)$ and $\{v_{n/2-1}, v_{n-1}, v_n\}$ induces K_3. See Figure 3.

For more relations between these graphs see Problem 9 in [12].

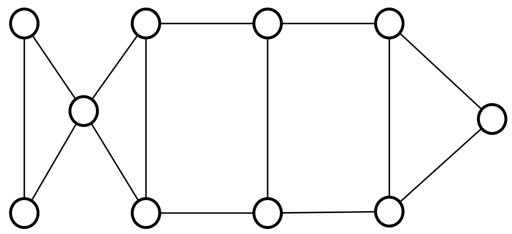

Figure 2: $\mathcal{F}(10)$ the fish on 10 vertices.

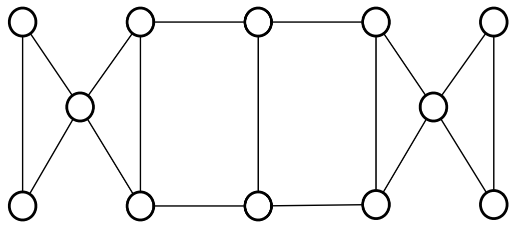

Figure 3: $\mathcal{C}(12)$ the candy on 12 vertices.

Definition 2.1 *Let $G = (V, E)$ be a graph with a triangle uvw such that u is a 2-vertex. $G^u = (V^u, E^u)$ is the graph obtained from G by contracting the triangle uvw in a vertex z^u with $N_{G^u}(z^u) = N_G(v) \cup N_G(w) - \{u, v, w\}$. Then, $V^u = (V - u - v - w) \cup \{z^u\}$, $|V^u| = n - 2$, $E^u = E - (E_G(\{u, v, w\}, V - \{u, v, w\}) \cup \{uv, vu, wv\}) \cup \{z^u v_i : v_i \in N_G(\{u, v, w\})\}$ and $|E^u| \leq |E| - 3$.*

Proposition 2.2 *For $n \geq 8, n$ even, $Exp(n) = \frac{3}{2}n - 1$.*

Proof: Let $n \geq 8$ be even. One can verify that $\mathcal{C}(n), \mathcal{H}(n), \mathcal{F}(n)$ have $\frac{3}{2}n - 1$ edges and are expandable.

Starting with $n = 8$ we give an inductive proof.

Let $n = 8$. We show that $Exp(8) = 11$. Suppose that $|E| \leq 10$. First we suppose that G contains two disjoint triangles $T_1 = \{v_1, v_2, v_3\}$ and $T_2 = \{v_4, v_5, v_6\}$. If $\delta(v_7) \geq 3$ then G is not 2-edge-connected whether $v_7v_8 \in E$ or $v_7v_8 \notin E$. So $\delta(v_7) = \delta(v_8) = 2$. If $v_7v_8 \notin E$ then $G - v_7 - v_8 = 2K_3$ thus G is not expandable. Then $v_7v_8 \in E$: w.l.o.g. $v_7v_1 \in E$ so by Property 2.1 $v_8v_1 \in E$ and then G cannot be 2-edge-connected. Thus any pair of triangles intersects.

We have $\Sigma_{i=1}^{8} \delta(v_i) \leq 20$, and, w.l.o.g., we can suppose that $\delta(v_i) = 2, 1 \leq i \leq 4$. First we suppose that $v_1v_2, v_3v_4 \in E$: from Property 2.2 v_1v_2 and v_3v_4 are contained in two triangles which intersect. So v_1, v_2, v_3, v_4 have a common neighbor; but then $\mu(G - v_1 - v_3) < n/2 - 1$.

Second we suppose that $v_1v_2 \in E$ is the unique edge connecting two 2-vertices. Let v_5 be the common neighbor of v_1, v_2. Since v_3, v_4 belongs to two different triangles and any pair of triangles intersect we have $v_3v_5, v_4v_5 \in E$. Since from Property 2.3 the second neighbors of v_3, v_4 are distinct we have w.l.o.g. $v_3v_6, v_4v_7 \in E$. Then from Property 2.2 $v_5v_6, v_5v_7 \in E$. This implies that $\delta(v_8) < 2$: a contradiction.

Finally, v_1, v_2, v_3, v_4 induce a stable set. Since the four vertices belong to four edge disjoint triangles, we have $|E| \geq 12$: a contradiction.

We suppose now that the proposition is true for $G \in MEG(n-2)$, $n \geq 10$: $Exp(n-2) = \frac{3}{2}(n-2) - 1$.

Let G be expandable with $|E| \leq \frac{3}{2}n - 1$, i.e. $\sum_{v \in V} \delta_G(v) \leq 3n - 2$. It follows from Property 2.1 that there is at least one 2-vertex u, so there are $v, w \in V$ such that $N_G(u) = \{v, w\}$ and from Property 2.2 $vw \in E$.

Claim 2.1 *If $G = (V, E)$ is expandable with $n \geq 8$, n even, and $|E| \leq \frac{3}{2}n - 1$ then $G^u = (V^u, E^u)$ is expandable and $|E^u| \leq \frac{3}{2}(n-2) - 1$.*

Proof: Let $x, y \in V^u$ with $xy \notin E^u$. There are two cases.

If $x, y \neq z^u$ then $x, y \in V$ and $xy \notin E$. Since G is expandable there is a matching M in $G - x - y$ with $|M| = \frac{n}{2} - 1$; w.l.o.g., $uv \in M$, $uw, vw \notin M$, and there is $t \in V - u - v$ s.t. $wt \in M$. So, $z^u t \in E^u$. Let $M^u = (M - uv - wt) \cup \{z^u t\}$: $|M^u| = |M| - 1 = \frac{n}{2} - 2$ and M^u is a perfect matching in $G^u - x - y$.

If $x = z^u$ and then $y \in V - u - v - w$, we have $vy \notin E$ and $wy \notin E$ (else $z^u y \in E^u$). Since G is expandable, there is a matching M in $G - v - y$ with $|M| = \frac{n}{2} - 1$ and $uw \in M$. Let $M^u = M - uw$: $|M^u| = |M| - 1 = \frac{n}{2} - 2$ and M^u is a perfect matching in $G^u - x - y$.

Therefore, G^u is expandable.

Since $|E| \leq \frac{3}{2}n - 1$ and $|E^u| \leq |E| - 3$, we have $|E^u| \leq \frac{3}{2}n - 4 = \frac{3}{2}(n-2) - 1$.
$\square$

Now let $G = (V, E) \in MEG(n), n \geq 10$: then $|E| \leq \frac{3}{2}n - 1$ and there is $u \in V$ with $N_G(u) = \{v, w\}$ and $vw \in E$. From Claim 2.1, G^u is expandable and $|E^u| \leq \frac{3}{2}(n-2) - 1$. By hypothesis, $Exp(n-2) = \frac{3}{2}(n-2) - 1$ so $|E^u| = \frac{3}{2}(n-2) - 1$ and $G^u \in MEG(n-2)$.

Finally, $|E| \geq |E^u| + 3 \Rightarrow |E| \geq \frac{3}{2}n - 1$. Since $|E| \leq \frac{3}{2}n - 1$ we have $Exp(n) = |E| = \frac{3}{2}n - 1$. $\square$

Property 2.4 *Let $G = (V, E) \in MEG(n), n \geq 10, n$ even. For a triangle uvw such that u is a 2-vertex we have $N_G(v) \cap N_G(w) = \{u\}$.*

Proof: This is a direct consequence of Proposition 2.2, else $|E| > |E^u| + 3 \geq \frac{3}{2}n - 1$. $\square$

Proposition 2.3 *For $n \geq 8, n$ even, $MEG(n) = \{\mathcal{C}(n), \mathcal{H}(n), \mathcal{F}(n)\}$.*

Proof: Starting with $n = 8$ we give an inductive proof.

Let $G = (V, E) \in MEG(8)$. We have $|E| = 11$, i.e. $\Sigma_{i=1}^{8} \delta(v_i) = 22$, since $\delta(G) \geq 2$, w.l.o.g., we can suppose that $\delta(v_1) = \delta(v_2) = 2$.

Claim 2.2 *Let $G = (V, E) \in MEG(8)$. If there are two 2-vertices u, v with $uv \notin E$ then $N_G(u) \cap N_G(v) = \emptyset$.*

Proof: Let $u = v_1, v = v_2$, we suppose that $v_1v_2 \notin E$ and $N_G(v_1) \cap N_G(v_2) \neq \emptyset$. It follows from Property 2.3 that $|N_G(v_1) \cap N_G(v_2)| = 1$; w.l.o.g. let $N_G(v_1) = \{v_3, v_4\}$ and $N_G(v_2) = \{v_3, v_5\}$. From Property 2.2 $v_3v_4, v_3v_5 \in E$, moreover $v_4v_5 \in E$ else G is not expandable; hence $G[\{v_1, \dots, v_5\}]$ has seven edges. From Property 2.1 we have $m_G(\{v_1, \dots, v_5\}, \{v_6, v_7, v_8\}) \geq 2$, but $\Sigma_{i=1}^{8} \delta(v_i) = 22$ implies $m_G(\{v_1, \dots, v_5\}, \{v_6, v_7, v_8\}) = 2$ and $\delta(v_i) = 2, i = 6, 7, 8$. Hence $G[\{v_6, v_7, v_8\}]$ has exactly two edges, w.l.o.g. v_6v_8, v_7v_8, so $v_6v_7 \notin E$, contradicting Property 2.2. □

Claim 2.3 *Let $G \in MEG(8)$. If there are two 2-vertices u, v with $uv \in E$ then $N_G(u) \cap N_G(v) = \{w\}, \delta(w) \geq 4$, and there is another 2-vertex $x \neq u, v$.*

Proof: From Property 2.2 uvw is a triangle of G, and from Property 2.1 $\delta(w) \geq 4$. Since $\Sigma_{i=1}^{8} \delta(v_i) = 22$ and $\delta(z) \geq 2$ for all vertices z there is $x \neq u, v$ such that $\delta(x) = 2$. □

We have the following cases:

- no two 2-vertices are linked: $v_1v_2 \notin E$, from Claim 2.2 we have $N_G(v_1) \cap N_G(v_2) = \emptyset$. W.l.o.g. we have $N_G(v_1) = \{v_3, v_4\}, N_G(v_2) = \{v_5, v_6\}, \delta(v_i) \geq 3, 3 \leq i \leq 6$; from Property 2.2 $v_3v_4, v_5v_6 \in E$ and by Claim 2.2 $\delta(v_7), \delta(v_8) \neq 2$. It follows that $\delta(v_i) = 3, 3 \leq i \leq 8$. If $v_3v_7, v_4v_7 \in E$ then $\mu(G - v_7 - v_2) < 3$. Thus, w.l.o.g. $N_G(v_7) = \{v_3, v_5, v_8\}$. Then $N_G(v_8) = \{v_4, v_6, v_7\}$, so $G = \mathcal{H}(8)$;
- two 2-vertices are linked: $v_1v_2 \in E$, from Claim 2.3 we have $v_1v_3, v_2v_3 \in E$, $\delta(v_3) \geq 4$ and there is a 2-vertex v_8. From Claim 2.2 $v_3 \notin N_G(v_8)$ so, w.l.o.g. $N_G(v_8) = \{v_6, v_7\}$ and $v_6v_7 \in E$ (Property 2.2). From Property 2.1 $m_G(\{v_6, v_7, v_8\}, V - \{v_6, v_7, v_8\}) \geq 2$.
  - one of v_6, v_7, say v_6, is a 2-vertex: if $\delta(v_4) = 2$ then $v_4v_3 \in E$ or $v_4v_7 \in E$ contradicting Claim 2.2. So $\delta(v_4) \geq 3$, and by symmetry $\delta(v_5) \geq 3$. Thus $\delta(v_4) = \delta(v_5) = 3$ and $G = \mathcal{C}(8)$;
  - both v_6, v_7 have degree at least three: as above $\delta(v_4), \delta(v_5) \neq 2$, so v_3 is a 4-vertex and $v_4, \dots, v_7$ are 3-vertices. Clearly $N_G(v_4) \neq \{v_3, v_6, v_7\}$ (otherwise there are three edges v_3v_5) so $v_4v_5 \in E$. Now $N_G(v_4) = \{v_5, v_6, v_7\}$ is impossible, so w.l.o.g., $N_G(v_4) = \{v_3, v_5, v_6\}$ and $N_G(v_5) = \{v_3, v_4, v_7\}$, and $G = \mathcal{F}(8)$.

We suppose now that $n \geq 10$ and $MEG(n-2) = \{\mathcal{C}(n-2), \mathcal{H}(n-2), \mathcal{F}(n-2)\}$.

We show that $MEG(n) = \{\mathcal{C}(n), \mathcal{H}(n), \mathcal{F}(n)\}$. Let $G = (V, E) \in MEG(n)$.
We have $\Sigma_{v \in V} \delta(v) = 3n - 2$, so at least two vertices have a degree 2. From Property 2.1 three of them cannot be connected together, else they induce a component C_3.

Let us consider the case where G contains two 2-vertices u, v such that $uv \in E$. From Property 2.2 we have $N_G(u) = \{v, w\}, N_G(v) = \{u, w\}$. We have $G - u - v = G^u$ with $z^u = w$ and thus $G^u \in \{\mathcal{C}(n-2), \mathcal{H}(n-2), \mathcal{F}(n-2)\}$.

Assume that $G^u = \mathcal{C}(n-2)$. First, suppose that $\delta_{G^u}(w) = 2$: In G, w has three neighbors of degree two u, v, x and one neighbor of degree four y; clearly, we have $\mu(G - u - y) < n/2 - 1$. Second, suppose that $\delta_{G^u}(w) = 4$: In G w, has four neighbors of degree two u, v, x, y; so $\mu(G - u - y) < n/2 - 1$. Actually, suppose that $\delta_{G^u}(w) = 3$. As shown by Figure 4, the vertex w has a neighbor x in G such that $\mu(G - u - x) < n/2 - 1$.

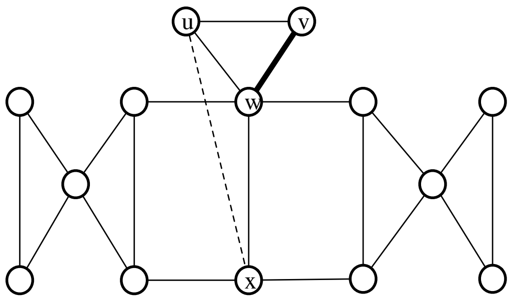

Figure 4: G is not expandable.

Now, assume that $G^u = \mathcal{H}(n-2)$. With the same argument as above the vertex w cannot have a degree three in G^u. Thus w is one of the two 2-vertices of G^u. Hence $G = \mathcal{F}(n)$.

Assume now that $G^u = \mathcal{F}(n-2)$. With the same kind of arguments as above w is the vertex of degree two in G^u which is not adjacent to another 2-vertex. Hence $G = \mathcal{C}(n)$.

From now on we consider the case where no two 2-vertices in G are adjacent.

First we prove that there is in G a triangle abc with $\delta_G(a) = 2$ and $\delta_G(b) = \delta_G(c) = 3$. From Property 2.2 there is a triangle uvw in G where u is a 2-vertex and $\delta_G(v), \delta_G(w) \geq 3$. If $\delta_G(v) = \delta_G(w) = 3$, uvw is the required triangle. Now assume that such a triangle does not exist, i.e. $\delta_G(v) > 3$ and $\delta_G(w) \geq 3$. From Property 2.4, we have $\delta_{G^u}(z^u) = \delta_G(v) + \delta_G(w) - 4 \geq 3$ and $\delta_G(x) = \delta_{G^u}(x)$ for all $x \in V - \{u, v, w\}$. From Claim 2.1, since $G \in MEG(n)$, $G^u \in MEG(n-2)$. If $G^u = \mathcal{F}(n-2)$ or $G^u = \mathcal{C}(n-2)$ then G^u has two adjacent vertices x and y with $\delta_{G^u}(x) = \delta_{G^u}(y) = 2$; since $\delta_{G^u}(z^u) \geq 3$, $x, y \neq z^u$; then x and y are also two adjacent 2-vertices in G: a contradiction. So $G^u = \mathcal{H}(n-2)$ which has two disjoint triangles abc and $a'b'c'$ verifying $\delta_{G^u}(a) = \delta_{G^u}(a') = 2$ and $\delta_{G^u}(b) = \delta_{G^u}(c) = \delta_{G^u}(b') = \delta_{G^u}(c') = 3$. Whatever z^u we choose, one of these two triangles, say abc, satisfies $\delta_G(a) = 2, \delta_G(b) = \delta_G(c) = 3$: a contradiction.

Now, consider G^a obtained by contracting abc in a vertex t. We have $\delta_{G^a}(t) = \delta_G(b) + \delta_G(c) - 4 = 2$, $G^a = \mathcal{H}(n-2)$ and t is one of its two 2-vertices. By expanding t in a triangle abc with $\delta_G(b) = \delta_G(c) = 3$, since $N_G(b) \cap N_G(c) = \{a\}$ by Property 2.4, we get $G = \mathcal{H}(n)$. $\square$

Finally we deal with the case $n \geq 9$ odd.

Proposition 2.4 *For $n \geq 9, n$ odd, $Exp(n) = n-1$ and $MEG(n) = \{C_{n-2} + K_2\}$.*

Proof: Let $G = (V, E) = C_{n-2} + K_2$. It is easy to verify that for any pair of vertices u, v with $uv \notin E$ we have $\mu(G-u-v) = \lfloor \frac{n}{2} \rfloor - 1$. So $Exp(n) \leq n-1$.

Assume that $G = (V, E) \in MEG(n)$ with $|E| \leq n-2$.

First, G has no 0-vertex. Indeed, if x is a 0-vertex then $G-x$ must be expandable. $G-x$ has an even number of vertices so from Proposition 2.2, we must have $|E| \geq \frac{3}{2}(n-1) - 1 > n-2$.

As a consequence in G there are at least four 1-vertices.

Two 1-vertices u and v cannot have a common neighbor w otherwise there exists a vertex w' such that $ww' \notin E$ so G is clearly not expandable. Now, let $k \geq 4$ be the number of 1-vertices. If u, v, w, z are 1-vertices with $uv, wz \in E$, considering $uw \notin E$ G is not expandable. In the case where u, v, w are 1-vertices, $uv \in E$, let z be the neighbor of w: considering $uz \notin E$ we see that G is not expandable. Thus the 1-vertices are not adjacent and their k neighbors induce a clique else G is not expandable. The remaining $n-2k$ vertices have a degree at least 2, so $|E| \geq k + \frac{k(k-1)}{2} + (n-2k) = \frac{1}{2}(k^2 - 3k + 2n) > n-2$: a contradiction.

Now let $G = (V, E) \in MEG(n)$; we prove that $G = C_{n-2} + K_2$. Using the same arguments as above G cannot have a 0-vertex and so it contains $k \geq 2$ 1-vertices and $|E| \geq \frac{1}{2}(k^2 - 3k + 2n)$. For $k \geq 3$, $\frac{1}{2}(k^2 - 3k + 2n) > n - 1 = Exp(n)$, so G has two 1-vertices and $n-2$ 2-vertices. So G consists of a $P_q, q \geq 2$, and disjoint cycles. None of the cycles can be even (if v_i, v_{i+2} are at distance two in an even cycle then $\mu(G - v_i - v_{i+2}) < \lfloor \frac{n}{2} \rfloor - 1$). If there are two odd cycles C_1, C_2 (and a P_q) by taking $u \in C_1$ and v a neighbor of an end vertex of P_q we have $\mu(G-u-v) < \lfloor \frac{n}{2} \rfloor - 1$. So G consists of a P_q and an odd cycle C_{n-q}. If $q \geq 3$, let u be an end vertex of P_q and v a vertex at distance two of u then $\mu(G-u-v) < \lfloor \frac{n}{2} \rfloor - 1$. Thus $G = C_{n-2} + K_2$. □

3 Minimal connected expandable graphs

In this section we will be interested in minimal expandable graphs which are connected and we will denote by $Cexp(n)$ the minimum number of edges of a connected MEG of order n, $CMEG(n)$ for short. We have $Cexp(n) \geq Exp(n)$ with equality in the case where $MEG(n)$ contains a connected element.

Let $n \geq 5$, the $(k, 2)$-pan is the graph with $n = k + 2$ vertices depicted as on Figure 5. It consists of a cycle of k vertices and a pending path of length 2. The (k, p)-pan are defined similarly with a pending path of length p. Note that $(k, 0)$-pan$= C_k$.

By $2K_3^+$ we denote the graph that consists of two copies of K_3 linked by one edge.

The problem being trivial for $n \leq 3$ we deal with the case $n \geq 4$.

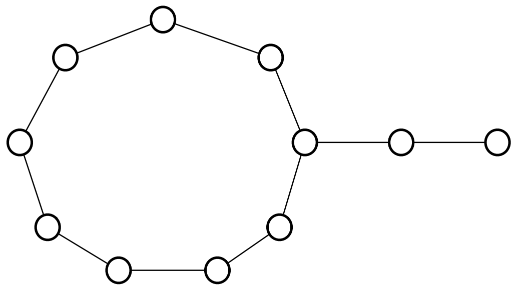

Figure 5: The $(9,2)$-pan.

Proposition 3.1

- *$Cexp(n) = n - 1$ and $CMEG(n) = \{K_{1,n-1}\}$, $4 \le n \le 5$;*
- *$Cexp(6) = 7$ and $CMEG(6) = \{2K_3^+\}$;*
- *$Cexp(n) = n$ and $CMEG(n) = \{C_n, (n-2,2) - pan\}$, $n \ge 7$, n odd;*
- *$Cexp(n) = Exp(n)$ and $CMEG(n) = MEG(n) = \{\mathcal{C}(n), \mathcal{H}(n), \mathcal{F}(n)\}$, $n \ge 8$, n even.*

Proof: For $n = 4$, since $MEG(4) = \{K_{1,3}, \bar{K}_{1,3}\}$ the proof is immediate.

Let $n = 5$. Since $MEG(5) = \{K_3 + 2K_1\}$ we have $Cexp(5) > Exp(5) = 3$. One can verify that $K_{1,4}$ is expandable. Thus a connected graph in MEG is necessarily a tree. One can easily verify that every tree of order 5 which is not $K_{1,4}$ is not expandable.

Let $n = 6$. Since $MEG(6) = \{2K_3\}$ and $2K_3$ is not connected we have $Cexp(6) > Exp(6) = 6$. Now since $2K_3$ is expandable, when connecting the two K_3's with an edge, the resulting connected graph $2K_3^+$ is expandable. So $Cexp(6) = 7$. Suppose that there exists $G = (V, E) \neq 2K_3^+$ such that $G \in CMEG(6)$. We have $1 \le \delta(G) \le 2$. First we assume that $\delta(G) = 1$. Let u be a 1-vertex in G, let v be its neighbor and let $v_i, 1 \le i \le 4$, the four other vertices of V. Necessarily $vv_i \in E, 1 \le i \le 4$, otherwise G is not expandable. There are exactly two edges of E in $G[\{v_1, v_2, v_3, v_4\}]$. W.l.o.g., these two edges are either v_1v_2, v_3v_4 or v_1v_2, v_1v_3. Now it is easy to verify that in both cases G is not expandable.

So $\delta(G) = 2$. Since $|E| = 7$, V contains a set $V^2 = \{v_1, v_2, v_3, v_4\}$ of 2-vertices.

(a) For $v \in V^2$, if $vw, vz \in E$ then $wz \in E$.

(b) No two edges in $G[V^2] = (V^2, E^2)$ can be adjacent: if $vw, vz \in E^2$ then $wz \in E^2$, from (a) this violates the connectivity of G.

It follows from (b) that $|E^2| \leq 2$.

If $E^2 = \emptyset$, then $v_i v_j \in E$ for all $v_i \in V^2$ and $j = 5, 6$ so $|E| \geq 8$: a contradiction.

Now $|E^2| \geq 1$ and w.l.o.g. $v_1 v_2 \in E, v_1 v_5, v_2 v_5 \in E$ from (a).

If $|E^2| = 1$ we have $v_i v_j \in E$ $(i = 3, 4; j = 5, 6)$ but then v_6 is a 2-vertex with $v_6 v_3, v_6 v_4 \in E$ and $v_3 v_4 \notin E$ which violates (a).

Finally, if $|E^2| = 2$, then $v_3 v_4 \in E^2$. Note that $|E| = 7$ implies $v_5 v_6 \in E$. So for $j = 5$ or $j = 6$, $v_3 v_j, v_4 v_j \in E$: for $j = 5$ we get $\delta(G) = 1$ (v_6 is a 1-vertex), and for $j = 6$ it gives $G = 2K_3^+$. Contradiction.

Let $n \geq 7$ be odd. Since $MEG(n) = \{C_{n-2} + K_2\}$ and $C_{n-2} + K_2$ is not connected we have $Cexp(n) > Exp(n) = n - 1$. One can verify that both C_n and $(n-2, 2)$-pan are expandable. So we have $Cexp(n) = n$. Suppose that there exists $G \in CMEG(n)$ and $G \neq C_n$, $G \neq (n-2, 2)$-pan. Since G is connected and has n edges, it contains exactly one (induced) cycle C. Now since $G \neq C_n$ it has at least one 1-vertex. Let u, v be two 1-vertices and u', v' be their neighbors, respectively. Since G is connected $u' \neq v, v' \neq u$. If $u' = v'$ then G is not expandable: u' must be universal, so there are exactly two 2-vertices and $n - 3 \geq 4$ 1-vertices. So $u' \neq v'$. Since G is expandable, $u'v'$ is an edge of G, and the neighbors of the 1-vertices are distinct and induce a clique. If G has at least four such vertices it follows that G has more than one cycle: a contradiction. If it has three, the cycle is a triangle and since $n \geq 7$ there must be another 1-vertex: a contradiction. So assume G has exactly two 1-vertices u, v: $uu', u'v', v'v \in E$. Assume $u'v' \notin C$; clearly one of u' and v', say v', has degree 2 (otherwise G has more than two 1-vertices). Then G is not expandable. So $u'v'$ is in C and it can be seen easily that G is not expandable (consider the non-edge $u'w$ where $N_G(u') \cap N_G(w) = \{z\}, z \neq v'$). Thus G has exactly one 1-vertex u, so G is a $(n-k, k)$-pan with $k > 0$. If $k = 1$ then G is not expandable (consider a non-edge between $u' \in C$ the neighbor of u and w a vertex at distance two of u'). Now, if $k \geq 3$ then G is not expandable (consider the non-edge between u' the neighbor of u and the vertex at distance two of u'). It follows that either $k = 0$ or $k = 2$: a contradiction.

For $n \geq 8$ even, the result holds since each graph of $MEG(n)$ is connected. □

4 Minimal completable graphs

In this section we consider the 'opposite' case of the previous one: instead of $d = 1$ in the general definition of Section 1 we will examine the case $d = \lfloor n/2 \rfloor - 1$.

We say that G is *completable* if for any matching $\bar{M}$ of $\bar{G}$ with $|\bar{M}| = \lfloor n/2 \rfloor - 1$ there exists an edge $uv \in G - V(\bar{M})$. By definition if $\bar{G}$ has no matching of size $\lfloor n/2 \rfloor - 1$ G is completable. The avoid this trivial situation we consider graphs G with a matching $\bar{M}$ of size $|\bar{M}| = \lfloor n/2 \rfloor - 1$ in $\bar{G}$.

As before, given n, we want to find a completable graph $G = (V, E)$ with n vertices and a minimum number $|E|$ of edges denoted by $Comp(n)$. Such a G will be called a *minimal completable graph* and the set of minimal completable graphs of order n is called $MCG(n)$.

The following statements are an immediate consequence of the definition of a completable graph.

Proposition 4.1 *If G=(V,E) is a graph with an even number of vertices then G is completable if and only if $\bar{G}$ has no perfect matching.*

Proposition 4.2 *If G=(V,E) is a graph with an odd number of vertices then G is completable if and only if $\bar{G}$ does not contain a matching M of size $\frac{1}{2}(n-1)-1$ and a triangle disjoint from $V(M)$.*

Proposition 4.3 *If $G=(V,E), |V| \geq 5$, is completable and has a vertex v of degree at most one then $G-v$ is completable.*

Proof: For n odd, assume $G-v$ is not completable; from Proposition 4.1, $\bar{G}-v$ has a perfect matching $\bar{M}$. There is an edge $zw \in \bar{M}$ such that v, w, z induce a triangle in $\bar{G}$. Considering the matching $\bar{M}' = \bar{M} - zw$ it follows from Proposition 4.2 that G is not completable.

For n even, assume $G-v$ is not completable; from Proposition 4.2, $\bar{G}-v$ has a matching $\bar{M}$ of size $\frac{1}{2}(n-2)-1$ plus a vertex disjoint triangle u, w, z. W.l.o.g. $vz \in \bar{G}$ and then $\bar{M} \cup \{uv, wz\}$ is a perfect matching of $\bar{G}$. From Proposition 4.1 G is not completable. □

Proposition 4.4 *For $4 \leq n \leq 7$ we have:*

- *$Comp(4) = Exp(4) = 3$ and $MCG(4) = MEG(4) = \{K_{1,3}, \bar{K}_{1,3}\}$;*
- *$Comp(5) = Exp(5) = 3$ and $MCG(5) = MEG(5) = \{K_3 + 2K_1\}$;*
- *$Comp(6) = 5$ and $MCG(6) = \{K_{1,5}\}$;*
- *$Comp(7) = 6$ and $MCG(7) = \{K_{1,6}, K_4 + 3K_1\}$.*

Proof: Let $4 \leq n \leq 5$; one observes that for these values a graph is completable if and only if it is expandable. The results follow from Proposition 2.1.

Let $n = 6$; clearly $K_{1,5}$ is completable. Thus $Comp(6) \leq 5$. Let c be the number of vertices in a largest connected component C of G. If $c \leq 3$ then $\mu(\bar{G}) = 3$ so G is not completable from Proposition 4.1. If $c = 4$ and $C = K_4$ then $|E| > 5$; if $c = 4$ and $C \neq K_4$ then $\mu(\bar{G}) = 3$. If $c = 5$: let $u \notin C$, so $\delta(u) = 0$; if $\mu(\bar{G}-u) = 2$ then $\mu(\bar{G}) = 3$; if $\mu(\bar{G}-u) \leq 1$ then $|E| \geq 6$. If $c = 6$ then G is connected; since $|E| \leq 5$ there is a 1-vertex v; its neighbor w must be universal, so $G = K_{1,5}$.

Let $n = 7$. One can verify that $K_{1,6}$ and $K_4 + 3K_1$ are completable. Suppose that $G = (V, E) \in MCG(7)$ with $|E| \leq 6$. If there is $u \in V$ with $N_G(u) = \{v\}$ then from Proposition 4.3 $G - u$ must be completable. Since $G - u$ has 6 vertices and at most 5 edges, from the above result we have $G - u = K_{1,5}$. Let $w \in V$ with $\delta_{G-u}(w) = 5$: if $v \neq w$ then $\mu(\bar{G}) = 3$ and G is not completable; if $v = w$, then $G = K_{1,6}$ and $|E| = 6$.

Assume now that G has no 1-vertex. Since $Comp(7) = |E| \leq 6$ there exists a 0-vertex u. Let c be the number of vertices in a largest connected component C of G. If $c \leq 3$ or $c = 6$ then $\mu(\bar{G}-u) = 3$ and since $\delta(u) = 0$ we have a matching of size two and a triangle in $\bar{G}$; so from Proposition 4.2 G is not completable. If $c = 4$ since $\delta(u) = 0$ and there is no 1-vertex, G consists of C and three 0-vertices. If $C \neq K_4$ then $\mu(\bar{G} - u) = 3$ and G is not completable. So $G = K_4 + 3K_1$ and $|E| = 6$. Now let $c = 5$: since $\delta(u) = 0$ and there is no 1-vertex, G consists of C and two 0-vertices u and v. If $\mu(\bar{G} - u - v) = 2$ then, from Proposition 4.2, G is not completable. Now assume $\mu(\bar{G} - u - v) \leq 1$; since $|E| \leq 6$ there are at least four edges in $\bar{G} - u - v$ which are adjacent to the same vertex so C is not connected: a contradiction. It follows that $Comp(7) = 6$ and $MCG(7) = \{K_{1,6}, K_4 + 3K_1\}$. $\square$

Proposition 4.5 *For $n \geq 8$, $Comp(n) = n - 1$ and $MCG(n) = \{K_{1,n-1}\}$.*

Proof: Clearly, $K_{1,n-1}$ is completable and $Comp(n) \leq n - 1$. The proof will be by induction on n and will need the following claim.

Claim 4.1 *For $n \geq 8$, if $G = (V, E)$ is completable with $|E| \leq n - 1$ then G has no 0-vertex.*

Proof: We will need the following result (see [3], chapter 7, page 120):

Theorem 4.6 *(Erdős, Gallai, 1959)*
Let G=(V,E) be a graph with m edges and let q be an integer. Then:

1. *For $\frac{1}{5}(2n + 1) \leq q < \frac{n}{2}$, if $m > q(2q + 1)$ then $\mu(G) > q$;*
2. *For $0 \leq q < \frac{1}{5}(2n + 1)$, if $m > \frac{1}{2}q(q - 1) + q(n - q)$ then $\mu(G) > q$.*

Let n be even. Assume there is a 0-vertex u. From Theorem 4.6 it exists $\bar{M} = \{a_i b_i : 1 \leq i \leq n/2 - 1\}$ a matching of $\bar{G}$ with $n/2 - 1$ edges. Let $G - V(\bar{M}) = \{v, w\}$. Since G is completable $vw \in E$ and $u \neq v, w$. We have $m_G(\{v, w\}, \{a_i, b_i\}) \geq 2$ for all $a_i b_i \in \bar{M}$, otherwise $\mu(\bar{G}) = n/2$. Thus $|E| \geq n - 1$, so $|E| = n - 1$ and then $m_G(\{v, w\}, \{a_i, b_i\}) = 2$ for all $a_i b_i \in \bar{M}$ and there is no edge between the four vertices a_i, b_i, a_j, b_j. W.l.o.g. let $u = a_1$. Since $\delta(u) = 0$ then $vb_1, wb_1 \in E$. Since $m_G(\{v, w\}, \{a_2, b_2\}) = 2$, we can assume $va_2 \notin E$. $\bar{M} - a_1b_1 - a_2b_2 \cup \{va_2, uw, b_1b_2\}$ is a perfect matching of $\bar{G}$ and G is not completable.

Let n be odd. Assume u is a 0-vertex. From Proposition 4.3, $G - u$ is completable, and from Proposition 4.1 $\mu(\bar{G} - u) < (n - 1)/2$. From Theorem 4.6 it exists $\bar{M} = \{a_i b_i : 1 \leq i \leq (n - 3)/2\}$ a matching of $\bar{G} - u$ with $(n - 3)/2$ edges. Let $G - u - V(\bar{M}) = \{v, w\}$. Since $G - u$ is completable $vw \in E$. We have $m_{G-u}(\{v, w\}, \{a_i, b_i\}) \geq 2$ for all $a_i b_i \in \bar{M}$, otherwise $\mu(\bar{G} - u) = (n - 1)/2$. Thus $|E| \geq n - 2$ and there are at most three edges of E between vw and $a_1 b_1$. W.l.o.g. $va_1 \notin E$. There must be a set $\{a_1, b_1, a_i, b_i\}$ with $i \neq 1$ and without edges inside, else $|E| \geq n$. Then $\bar{M} - a_1b_1 - a_ib_i \cup \{uw, va_1\}$ and a_i, b_1, b_i form a matching and a vertex disjoint triangle spanning V and it follows that $|E| \geq n$. $\square$

Let $G \in MCG(n)$ and $n \geq 8$.

From Claim 4.1, since $Comp(n) \leq n - 1$, there is a 1-vertex u in G. From Proposition 4.3, the graph $G - u$ is completable.

For $n = 8$, we have $Comp(8) \leq 7$. Since $G - u$ is completable and has at most 6 edges, from Proposition 4.4, $G - u$ has 6 edges and is either $K_4 + 3K_1$ or $K_{1,6}$. In the first case, G would have a 0-vertex contradicting the Claim 4.1. In the second case, let w be the center of $K_{1,6}$: if u is not linked to w then G is not completable thus $uw \in E$ and so we get $MCG(8) = \{K_{1,7}\}$.

For $n > 8$, we assume that $MCG(n-1) = \{K_{1,n-2}\}$. $G - u$ is completable and has at most $n-2$ edges, so from our assumption $G-u$ has $n-2$ edges and is $K_{1,n-2}$ To show the result, we notice that, as above, u is linked to the center of $K_{1,n-2}$ and we get $MCG(n) = \{K_{1,n-1}\}$ and $Comp(n) = n - 1$. □

If we are interested in minimal completable graphs which are connected the only non trivial case to consider is for $n = 5$ (for other values of n there are minimal connected expandable graphs). For $n = 5$ there is only one minimal connected expandable graph $K_{1,4}$.

5 Conclusion

For $d = 1$ and $d = \lfloor n/2 \rfloor - 1$ we have determined a minimal graph G such that to any matching $\bar{M}$ of d edges of $\bar{G}$ one can associate a matching of $\lfloor n/2 \rfloor - d$ edges in $G - V(\bar{M})$. It would be interesting to consider other values of d.

A generalization of this problem would be to introduce weights on the edges. The goal would be to find minimal or maximal weighted graphs in $MEG(n)$ or $MCG(n)$. A straightforward corollary of Proposition 2.4 is that the problem with weights is $\mathcal{NP}$-hard (since $MEG(n) = \{C_{n-2} + P_2\}$ for n odd, a direct reduction from Traveling Salesman Problem [6] gives the result).

In connection with matching extensions, the class of (p, q, r)-graphs has been studied in [2, 7, 8]: these are graphs such that when deleting any set of p vertices the remaining graph $G' = (V', E')$ has matchings of size q and any such matching can be extended to a matching of size $(|V'| - r)/2$. In our paper we have considered that in a complete graph $K_n = (V, E)$ on n vertices we had to find a smallest possible subset $F \subseteq E$ such that any matching $\bar{M}$ of size d in $\bar{G} = (V, E - F)$ can be extended to a (almost)-perfect matching by adding edges of $G = (V, F)$. A natural generalization would be to consider an arbitrary graph instead of K_n; this would amount to forbidding the use of some edges in the matchings.

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