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Some recent results on the design and implementation of interval observers for uncertain systems

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Abstract: Based on the theory of positive systems, the goal of interval observers is to compute sets of admissible values of the state vector at each instant of time for systems subject to bounded uncertainties (noises, disturbances and parameters). The size of the estimated sets, which should be minimised, has to be proportional to the model uncertainties. An interval estimation can be seen as a conventional point estimation (the centre of the interval) with an estimation error given by the interval radius. The reliable uncertainties propgation performed in this context can be useful in several fields such as robust control, diagnosis and fault-tolerant control. This paper presents some recent results on interval observers for several dynamical systems classes such as continuous-time and switched systems.

Keywords: Interval observer, robustness, continuous-time, robust control, uncertainties

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1 Introduction

The problem of state vector estimation is very challenging and can be encountered in many applications [1, 2, 3]. For linear time-invariant models there are plenty solutions, among them the most popular are the Luenberger observer and Kalman filter for deterministic and stochastic settings, respectively. Among other popular solutions for estimation, it is worth to mention high-gain observers [4] and high order sliding mode observers/differentiators [5]. In the nonlinear case, observers or controllers design is based on transformations of the system into a canonical form (frequently close to a linear canonical representation [1, 2, 3]). Since such a transformation may depend on uncertain parameters or may be unknown due to the model complexity, then the application of a transformation may be an obstruction in practice. The reader can refer for instance to the survey [6] on the application of different observers for control and estimation of nonlinear systems. In this context, the class of Linear Parameter-Varying (LPV) systems became very popular in applications: a wide class of nonlinear systems can be presented in an LPV form (in this case the system equations are extended). A partial linearity of LPV models allows a rich spectrum of methods developed for linear systems to be applied [7, 8, 9, 10].

In several fields, such as networked control systems, electrical devices/circuits and congestion modeling [11], switched systems appear as interesting modeling tools, that can be more accurate than LPV models, to take into account some complex physical behaviours. These systems are one of the most important classes of Hybrid Dynamical Systems (HDS). They consist of a family of continuous and discrete-time subsystems and a switching rule orchestrating among them. At each time, only one subsystem is active. Several works treated switched systems such as [12], [11].

In addition to models complexity, another difficulty for an estimator design consists in the model uncertainty (unknown parameters or/and external disturbances). In the presence of uncertainties, the design of a conventional estimator, converging to the ideal value of the state, may be hard to be realized. Application of sliding-mode tools [13] or other disturbances cancellation approaches may

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solve this issue in some cases, however, in general, in the presence of uncertainties, the state estimation error is never approaching zero (it can be bounded or asymptotically bounded, and different versions of practical stability are used for analysis). In this case an interval estimation may still remain feasible: an observer can be constructed that, using input-output information, evaluates the set of admissible values (interval) for the state at each instant of time. The interval width has to be minimized by tuning the observer parameters, and should be proportional to the size of the model uncertainties. Despite such a formulation looks like a simplification of the state estimation problem (instead of the actual state, an interval is estimated), in fact it is an improvement since the interval centre can be used as a state pointwise estimate, while the interval width gives the admissible deviation from that value. Thus, an interval estimator provides an accuracy evaluation for bounded uncertainties that may not have known statistics [14, 15, 16, 17, 18]. This idea has been initially proposed in [19] and recently extended to several classes of dynamical systems.

This paper constitues a non-exhaustive overview of recent results on interval observers which form a subclass of set-membership estimators and whose design is based on the monotone systems theory [16, 18, 20, 21, 22]. Only the case of continuous-times systems is considered, the reader can refer for instance to [23, 24, 25, 26, 27, 28] for the case of discrete-time systems.

In such a way the main restriction for interval observers design consists in providing cooperativity of the interval estimation error dynamics by a proper design. Such a complexity has been overcame in [29, 21, 30] for Linear Time-Invariant (LTI) systems and extended to Linear Time-Varying (LTV), Linear Parameter-Varying and some particular classes of nonlinear systems. In those studies, it has been shown that under some mild conditions, by applying a similarity transformation, a Hurwitz matrix could be transformed to a Hurwitz and Metzler one. In the following, the main ideas of interval observers design are explained for different classes of systems.

2 Preliminaries

Real and integer numbers sets are denoted by $\mathbb{R}$ and $\mathbb{Z}$ respectively, $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$. The Euclidean norm for a vector $x \in \mathbb{R}^n$ will be denoted as $|x|$, and for a measurable and locally essentially bounded input $u : \mathbb{R}_+ \to \mathbb{R}$, the symbol $||u||_{[t_0,t_1]}$ denotes its L_∞ norm $||u||_{[t_0,t_1]} = ess\sup\{|u(t)|, t \in [t_0, t_1]\}$. If $t_1 = +\infty$, then we will simply write $||u||$. We will denote as $\mathcal{L}_\infty$ the set of all inputs u with the property $||u|| < \infty$. Denote the sequence of integers $1, ..., k$ as $\overline{1,k}$. The symbols I_n, $E_{n\times m}$ and E_p are the identity matrix with dimension $n \times n$ and the matrix with all elements equal 1 with dimensions $n \times m$ and $p \times 1$, respectively. Finally, denote by $||A||_2$ the induced L_2 matrix norm.

2.1 Interval relations

For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood elementwise. The relation $P \prec 0$ ($P \succ 0$) means that the matrix $P \in \mathbb{R}^{n\times n}$ is negative (positive) definite. Given a matrix $A \in \mathbb{R}^{m\times n}$, define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ (similarly for vectors) and denote the absolute value of a matrix by $|A| = A^+ + A^-$.

Lemma 1. [22] *Let $x \in \mathbb{R}^n$ be a vector satisfying $\underline{x} \leq x \leq \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$.*

(1) *If $A \in \mathbb{R}^{m\times n}$ is a constant matrix, then*

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}. \quad (1)$$

(2) *If $A \in \mathbb{R}^{m\times n}$ is an uncertain matrix verifying $\underline{A} \leq A \leq \overline{A}$ for some $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, then*

$$\begin{aligned} \underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- &\leq Ax \\ \leq \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^-. \end{aligned} \quad (2)$$

Furthermore, if $-\overline{A} = \underline{A} \leq 0 \leq \overline{A}$, then the inequality (2) can be simplified: $-\overline{A}(\overline{x}^+ + \underline{x}^-) \leq Ax \leq \overline{A}(\overline{x}^+ + \underline{x}^-)$.

2.2 Nonnegative continuous-time linear systems

A matrix $A \in \mathbb{R}^{n\times n}$ is called Hurwitz if all its eigenvalues have negative real parts, it is called Metzler if all its elements outside the main diagonal are nonnegative. Any solution of the linear system

$$\begin{aligned} \dot{x} &= Ax + B\omega(t),\ \omega : \mathbb{R}_+ \to \mathbb{R}^q_+,\ \omega \in \mathcal{L}^q_\infty, \\ y &= Cx + D\omega(t), \end{aligned} \quad (3)$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and a Metzler matrix $A \in \mathbb{R}^{n\times n}$, is elementwise nonnegative for all $t \geq 0$ provided that $x(0) \geq 0$ and $B \in \mathbb{R}^{n\times q}_+$ [31, 32]. The output solution $y(t)$ is nonnegative if $C \in \mathbb{R}^{p\times n}_+$ and $D \in \mathbb{R}^{p\times q}_+$. Such dynamical systems are called cooperative (monotone) or nonnegative if only initial conditions in $\mathbb{R}^n_+$ are considered

[31, 32]. The stability of a Metzler matrix $A \in \mathbb{R}^{n\times n}$ can be checked verifying a Linear Programming (LP) problem

$$A^{\mathrm{T}}\lambda < 0$$

for some $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$, or a Lyapunov matrix equation

$$A^{\mathrm{T}}P + PA \prec 0$$

for a diagonal matrix $P \in \mathbb{R}^{n\times n}$, $P > 0$. The L_1 and L_∞ gains for nonnegative systems (3), *i.e.* gains of transfer function from input to output in different norms, have been studied in [33, 34], for this kind of systems these gains are interrelated.

Lemma 2. [33, 34] *Let the system (3) be nonnegative (i.e. A is Metzler, $B \ge 0$, $C \ge 0$ and $D \ge 0$), then it is asymptotically stable if and only if there exist $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ and a scalar $\gamma > 0$ such that the following LP problem is feasible:*

$$\begin{bmatrix} A^T\lambda + C^T E_p \\ B^T\lambda - \gamma E_q + D^T E_p \end{bmatrix} < 0.$$

Moreover, in this case, the L_1 gain of the operator $\omega \to y$ is lower than γ.

Lemma 3. [33, 34] *Let the system (3) be nonnegative (i.e. A is Metzler, $B \ge 0$, $C \ge 0$ and $D \ge 0$), then it is asymptotically stable if and only if there exist $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ and a scalar $\gamma > 0$ such that the following LP problem is feasible:*

$$\begin{bmatrix} A\lambda + BE_q \\ C\lambda - \gamma E_p + DE_q \end{bmatrix} < 0.$$

Moreover, in this case, the L_∞ gain of the transfer $\omega \to y$ is lower than γ.

The conventional results and definitions on L_2/L_∞ stability for linear systems can be found in [35].

2.3 Nonnegative systems with delays

Consider a linear system with constant delays

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{N} A_i x(t-\tau_i) + \omega(t), \qquad (4)$$

where $x(t) \in \mathbb{R}^n$ is the state, $x_t \in \mathcal{C}^n_\tau$ for $\tau = \max_{1\le i\le N}\tau_i$ where $\tau_i \in \mathbb{R}_+$ are the delays ($\mathcal{C}_\tau = C([-\tau,0],\mathbb{R})$ is the set of continuous maps from $[-\tau,0]$ into $\mathbb{R}$, $\mathcal{C}_{\tau+} = \{y \in \mathcal{C}_\tau : y(s) \in \mathbb{R}_+,\ s \in [-\tau,0]\}$); a piecewise continuous function $\omega \in \mathcal{L}^n_\infty$ is the input; the constant matrices A_i, $i = \overline{0,N}$ have appropriate dimensions. The system (4) is called cooperative or nonnegative if it admits $x(t) \in \mathbb{R}^n_+$ for all $t \ge t_0$ provided that $x_{t_0} \in \mathcal{C}^n_{\tau+}$ and $\omega : \mathbb{R} \to \mathbb{R}^n_+$ [36].

Lemma 4. [37, 36] *The system (4) is nonnegative for all $\tau \in \mathbb{R}_+$ iff A_0 is Metzler and A_i, $i = \overline{1,N}$ are nonnegative matrices.*

3 Interval observers for LTI systems

Let us start the line of design of interval observers with the simplest cases of time-invariant linear models.

Consider the following system

$$\dot{x}(t) = Ax(t) + d(t),\ y(t) = Cx(t) + v(t),\ t \in \mathbb{R}_+ \qquad (5)$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ is the output; $d(t) \in \mathbb{R}^n$ is the disturbance, $d \in \mathcal{L}^n_\infty$; $v(t) \in \mathbb{R}^p$ is the measurement noise, $v \in \mathcal{L}^p_\infty$; the matrices A, C have appropriate dimensions. This system has three sources of uncertainties: initial conditions for $x(0)$ and instant values of d and v. It is assumed that all these uncertain factors belong to known intervals.

Assumption 1. *Let $x(0) \in [\underline{x}_0, \overline{x}_0]$ for some known $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$; let also two functions $\underline{d}, \overline{d} \in \mathcal{L}^n_\infty$ and a constant $V > 0$ be given such that*

$$\underline{d}(t) \le d(t) \le \overline{d}(t),\ -VE_p \le v(t) \le VE_p \quad \forall t \ge 0.$$

Using the available information, the goal is to calculate two bounds $\underline{x}, \overline{x} \in \mathcal{L}^n_\infty$ verifying

$$\underline{x}(t) \le x(t) \le \overline{x}(t) \quad \forall t \ge 0. \qquad (6)$$

An interval observer, composed of two conventional ones, is a solution to this problem:

$$\begin{cases} \dot{\underline{x}}(t) &= A\underline{x}(t) + L[y(t) - C\underline{x}(t)] - |L|E_pV + \underline{d}(t), \\ \dot{\overline{x}}(t) &= A\overline{x}(t) + L[y(t) - C\overline{x}(t)] + |L|E_pV + \overline{d}(t), \\ & \underline{x}(0) = \underline{x}_0,\ \overline{x}(0) = \overline{x}_0, \end{cases} \qquad (7)$$

where $L \in \mathbb{R}^{n\times p}$ is the observer gain to be designed. The conditions to satisfy for L are given in Theorem 1.

Theorem 1. [19] *Let Assumption 1 hold, the matrix $A - LC$ be Metzler and $x \in \mathcal{L}^n_\infty$, then the solutions of (5) and (7) satisfy the relations (6). In addition, $\underline{x}, \overline{x} \in \mathcal{L}^n_\infty$ if $A - LC$ is Hurwitz.*

In the following only the proof of this theorem is given as an illustration.

Proof. Define two estimation errors

$$\underline{e}(t) = x(t) - \underline{x}(t),\ \overline{e}(t) = \overline{x}(t) - x(t),$$

which yield differential equations:

$$\begin{aligned}\dot{\underline{e}}(t) &= [A - LC]\underline{e}(t) + Lv(t) + |L|E_pV + d(t) - \underline{d}(t),\\ \dot{\overline{e}}(t) &= [A - LC]\overline{e}(t) - Lv(t) + |L|E_pV + \overline{d}(t) - d(t).\end{aligned}$$

By Assumption 1, for all $t \geq 0$

$$|L|E_pV \pm Lv(t) \geq 0,\ d(t) - \underline{d}(t) \geq 0,\ \overline{d}(t) - d(t) \geq 0.$$

Since all inputs of $\underline{e}(t)$, $\overline{e}(t)$ are positive and $\underline{e}(0) \geq 0$, $\overline{e}(t) \geq 0$, if $A - LC$ is a Metzler matrix, then $\underline{e}(t) \geq 0$, $\overline{e}(t) \geq 0$ for all $t \geq 0$ [31, 32]. The property (6) follows from these relations. Similarly, since all inputs of $\underline{e}(t)$, $\overline{e}(t)$ are bounded, if $A - LC$ is Hurwitz, then $\underline{e}, \overline{e} \in \mathcal{L}_\infty^n$ and the boundedness of $\underline{x}, \overline{x}$ is implied by the same property of x. □

From this proof we can conclude that the idea to design an interval observer is to guarantee the nonnegativity of the estimation error dynamics. Therefore, the observer gain L has to be designed such that the matrix $A - LC$ is Metzler and Hurwitz. In addition, in order to optimize the width of the estimated interval $[\underline{x}(t), \overline{x}(t)]$ the problem of L_1 or L_∞ optimization of that gain value can be posed. Using lemmas 2 and 3 this problem can be formulated as a LP computational procedure. In the case of a L_1 optimization, it is necessary to find $\lambda \in \mathbb{R}^n$, $w \in \mathbb{R}^p$ and a diagonal matrix $M \in \mathbb{R}^{n\times n}$ such that

$$\begin{gathered}\begin{bmatrix} A^{\mathrm{T}}\lambda - C^{\mathrm{T}}w + E_n \\ \lambda - \gamma E_n \end{bmatrix} < 0,\\ \lambda > 0,\ M \geq 0, \\ A^{\mathrm{T}}\lambda - C^{\mathrm{T}}w + M\lambda \geq 0,\end{gathered} \tag{8}$$

then $w = L^{\mathrm{T}}\lambda$ [38]. The reader can also refer to [39] for another LP problem formulation and solution for a more generic LPV system, and to [40] for a H_∞ oprimization procedure.

Formulation (8) provides an effective computational tool to design interval observers, however, it is only a sufficient condition and in some cases, this LP problem may have no solution, but it does not imply that it is not possible to design an interval observer. Roughly speaking in this case it is not possible to find L such that $A - LC$ is simultaneously Metzler and Hurwitz. However, it is well known that Hurwitz property of matrices is preserved under similarities transformation of coordinates. Then, to overcome this issue, it is possible to design a gain L such that the matrix $A - LC$ is Hurwitz and next to find a nonsingular matrix $S \in \mathbb{R}^{n\times n}$ such that, in the new coordinates $z = Sx$, the state matrix $D = S(A - LC)S^{-1}$ is Metzler (it is Hurwitz by construction). The conditions of existence of such a real transformation matrix S are given in the following lemma.

Lemma 5. [21] *Given the matrices $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$. If there is a matrix $L \in \mathbb{R}^{n\times p}$ such that $A - LC$ and D have the same eigenvalues, then there exists a matrix $S \in \mathbb{R}^{n\times n}$ such that $D = S(A - LC)S^{-1}$ provided that the pairs $(A - LC, \chi_1)$ and (D, χ_2) are observable for some $\chi_1 \in \mathbb{R}^{1\times n}$, $\chi_2 \in \mathbb{R}^{1\times n}$.*

Note that (5) can be rewritten as follows:

$$\dot{x}(t) = (A - LC)x(t) + Ly(t) - Lv(t) + d(t).$$

Under the conditions of this lemma and in the new coordinates $z = Sx$, the system (5) takes the form:

$$\dot{z}(t) = Dz(t) + SLy(t) + \delta(t),\ \delta(t) = S[d(t) - Lv(t)]. \tag{9}$$

And using Lemma 1 we obtain that $\underline{\delta}(t) \leq \delta(t) \leq \overline{\delta}(t)$, where $\underline{\delta}(t) = S^+\underline{d}(t) - S^-\overline{d}(t) - |SL|E_pV$ and $\overline{\delta}(t) = S^+\overline{d}(t) - S^-\underline{d}(t) + |SL|E_pV$. For the system (9) all conditions of Theorem 1 are satisfied and an interval observer similar to (7) can be designed:

$$\begin{aligned}\dot{\underline{z}}(t) &= D\underline{z}(t) + SLy(t) + \underline{\delta}(t),\\ \dot{\overline{z}}(t) &= D\overline{z}(t) + SLy(t) + \overline{\delta}(t),\\ \underline{z}(0) = S^+\underline{x}_0 - S^-\overline{x}_0,&\ \overline{z}(0) = S^+\overline{x}_0 - S^-\underline{x}_0,\\ \underline{x}(t) &= (S^{-1})^+\underline{z}(t) - (S^{-1})^-\overline{z}(t),\\ \overline{x}(t) &= (S^{-1})^+\overline{z}(t) - (S^{-1})^-\underline{z}(t),\end{aligned} \tag{10}$$

where the relations (1) are used to calculate the initial conditions for $\underline{z}, \overline{z}$ and the estimates $\underline{x}, \overline{x}$. It is easy to show that in (10) the inclusion (6) is satisfied and $\underline{x}, \overline{x} \in \mathcal{L}_\infty^n$.

If conditions of Lemma 5 are not satisfied, then it is possible also to find a time-varying change of coordinates [29, 30], for instance, based on the Jordan canonical form, as in the next lemma.

Lemma 6. [29] *Let $A - LC$ be Hurwitz, then there exists an invertible matrix function $P : \mathbb{R} \to \mathbb{R}^{n\times n}$, of class C^∞ elementwise, $||P(t)||_2 < +\infty$ for all $t \in \mathbb{R}$, such that for all $t \in \mathbb{R}$*

$$\dot{P}(t) = DP(t) - P(t)(A - LC),$$

where $D \in \mathbb{R}^{n\times n}$ is a Hurwitz and Metzler matrix.

4 Interval observers for time-delay systems

Consider the system (4) with an output $y \in \mathbb{R}^p$ subject to a bounded noise $v \in \mathcal{L}_\infty^p$:

$$y = Cx, \quad \psi = y + v(t), \tag{11}$$

where $C \in \mathbb{R}^{p\times n}$. Below the relation $a \le b$ for $a, b \in \mathcal{C}_\tau^n$ is understood in the sense that $a(\theta) \le b(\theta)$ for all $\theta \in [-\tau, 0]$.

Assumption 2. *Let $x \in \mathcal{L}_\infty^n$ with $\underline{x}_0 \le x_{t_0} \le \overline{x}_0$ for some $\underline{x}_0, \overline{x}_0 \in \mathcal{C}_\tau^n$; $||v|| \le V$ for a given $V > 0$; and $\underline{\omega}(t) \le \omega(t) \le \overline{\omega}(t)$ for all $t \ge t_0$ for some known $\underline{\omega}, \overline{\omega} \in \mathcal{L}_\infty^n$.*

In this assumption it is supposed that the state of the system (4) is bounded with an unknown upper bound, but with a specified admissible set for initial conditions $[\underline{x}_0, \overline{x}_0]$. The uncertainties of the system are collected in the external input ω with known bounds $\underline{\omega}, \overline{\omega}$, in the measurement noise v and in the initial conditions $[\underline{x}_0, \overline{x}_0]$.

As it is shown above, interval observers have an enlarged dimension (the examples given above have $2n$ variables to estimate n states). Thus, design of reduced order interval observers is of a big importance for applications. A reduced order interval observer for time-delay system (4), (11) has been proposed in [41, 42], those ideas are explained below.

For the system (4), (11) there exists a nonsingular matrix $S \in \mathbb{R}^{n\times n}$ such that $x = S\,[y^{\mathrm{T}}\ z^{\mathrm{T}}]^{\mathrm{T}}$ for an auxiliary variable $z \in \mathbb{R}^{n-p}$ (define $S^{-1} = [C^{\mathrm{T}}\ Z^{\mathrm{T}}]^{\mathrm{T}}$ for a matrix $Z \in \mathbb{R}^{(n-p)\times n}$), then

$$\begin{aligned} \dot{y}(t) &= R_1 y(t) + R_2 z(t) + \textstyle\sum_{i=1}^N [D_{1i} y(t-\tau_i) \\ &\quad + D_{2i} z(t-\tau_i)] + C\omega(t), \\ \dot{z}(t) &= R_3 y(t) + R_4 z(t) + \textstyle\sum_{i=1}^N [D_{3i} y(t-\tau_i) \\ &\quad + D_{4i} z(t-\tau_i)] + Z\omega(t) \end{aligned} \tag{12}$$

for some matrices R_k, D_{ki}, $k = \overline{1,4}$, $i = \overline{1,N}$ of appropriate dimensions. Introducing a new variable $w = z - Ky = Ux$ for a matrix $K \in \mathbb{R}^{(n-p)\times p}$ with $U = Z - KC$, from (12) the following equation is obtained

$$\begin{aligned} \dot{w}(t) &= G_0\psi(t) + M_0 w(t) + \sum_{i=1}^N [G_i\psi(t-\tau_i) + M_i w(t-\tau_i)] \\ &\quad + \beta(t),\ \beta(t) = U\omega(t) - G_0 v(t) - \sum_{i=1}^N G_i v(t-\tau_i), \end{aligned} \tag{13}$$

where $\psi(t)$ is defined in (11), $G_0 = R_3 - KR_1 + (R_4 - KR_2)K$, $M_0 = R_4 - KR_2$, and $G_i = D_{3i} - KD_{1i} + \{D_{4i} - KD_{2i}\}K$, $M_i = D_{4i} - KD_{2i}$ for $i = \overline{1,N}$. Under Assumption 2 and using the relations (1) the following inequalities follow:

$$\begin{aligned} &\underline{\beta}(t) \le \beta(t) \le \overline{\beta}(t), \\ &\underline{\beta}(t) = U^+\underline{\omega}(t) - U^-\overline{\omega}(t) - \sum_{i=0}^N |G_i| E_p V, \\ &\overline{\beta}(t) = U^+\overline{\omega}(t) - U^-\underline{\omega}(t) + \sum_{i=0}^N |G_i| E_p V. \end{aligned}$$

Then, a reduced order interval observer can be proposed for (4):

$$\begin{aligned} \dot{\underline{w}}(t) &= G_0\psi(t) + M_0\underline{w}(t) + \sum_{i=1}^N [G_i\psi(t-\tau_i) + M_i\underline{w}(t-\tau_i)] + \underline{\beta}(t), \\ \dot{\overline{w}}(t) &= G_0\psi(t) + M_0\overline{w}(t) + \sum_{i=1}^N [G_i\psi(t-\tau_i) + M_i\overline{w}(t-\tau_i)] + \overline{\beta}(t), \\ \underline{w}_0 &= U^+\underline{x}_0 - U^-\overline{x}_0,\ \overline{w}_0 = U^+\overline{x}_0 - U^-\underline{x}_0. \end{aligned} \tag{14}$$

The applicability conditions for (14) are given below.

Theorem 2. [41] *Let Assumption 2 be satisfied and the matrices M_0, M_i, $i = \overline{1,N}$ form an asymptotically stable cooperative system. Then $\underline{x}, \overline{x} \in \mathcal{L}_\infty^n$ and*

$$\underline{x}(t) \le x(t) \le \overline{x}(t) \quad \forall t \ge 0,$$

where

$$\begin{aligned} \underline{x}(t) &= S^+[\underline{y}(t)^T\,\underline{z}(t)^T]^T - S^-[\overline{y}(t)^T\,\overline{z}(t)^T]^T, \\ \overline{x}(t) &= S^+[\overline{y}(t)^T\,\overline{z}(t)^T]^T - S^-[\underline{y}(t)^T\,\underline{z}(t)^T]^T, \end{aligned} \tag{15}$$

$$\begin{aligned} &\underline{y}(t) = \psi(t) - V,\ \overline{y}(t) = \psi(t) + V, \\ &\underline{z}(t) = \underline{w}(t) + K^+\underline{y} - K^-\overline{y},\ \overline{z}(t) = \overline{w}(t) + K^+\overline{y} - K^-\underline{y}. \end{aligned} \tag{16}$$

The main condition of Theorem 2 is rather straightforward: the matrices M_0, M_i, $i = \overline{1,N}$ have to form a stable cooperative system. It is a standard LMI problem to find a matrix K such that the system composed by M_0, M_i, $i = \overline{1,N}$ is stable, but to find a matrix K making the system stable and cooperative simultaneously could be more complicated. However, the advantage of Theorem 2 is that its main condition can be reformulated using LMIs following the idea of [43]. If the LMIs given in [41] are not satisfied, the assumption that the matrix M_0 is Metzler and the matrices M_i, $i = \overline{1,N}$ are nonnegative can be relaxed using Lemma 5. This approach can also be applied to the system (4) with time-varying and uncertain delays [41]. The cases of delayed measurements are studied in [44, 42]. An extension to descriptor delay systems is presented in [45].

5 The case of LPV models

Consider an LPV system:

$$\dot{x} = [A_0 + \Delta A(\rho(t))]x + d(t),\ y = Cx + v(t), \quad (17)$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^p$ is the output available for measurements, $\rho(t) \in \Pi \subset \mathbb{R}^r$ is the vector of scheduling parameters, $\rho \in \mathcal{L}_\infty^r$. The values of the scheduling vector ρ are not available for measurements, and only the set Π of admissible values is known. The matrices $A_0 \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$ are known, the matrix function $\Delta A : \Pi \to \mathbb{R}^{n\times n}$ is piecewise continuous. Therefore, the results of this section are given for the worst case where the dependency between the system parameters are not assumed to be available. The signals $d : \mathbb{R}_+ \to \mathbb{R}^n$ and $v : \mathbb{R}_+ \to \mathbb{R}^p$ are the disturbances and measurement noise respectively.

Assumption 3. *Let $x \in \mathcal{L}_\infty^n$ and $x(0) \in [\underline{x}_0, \overline{x}_0]$ for some known $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$; $\underline{d}(t) \le d(t) \le \overline{d}(t)$ and $-VE_p \le v(t) \le VE_p$ for all $t \ge 0$ and some known $\underline{d}, \overline{d} \in \mathcal{L}_\infty^n$ and $V > 0$; $\underline{\Delta A} \le \Delta A(\rho) \le \overline{\Delta A}$ for all $\rho \in \Pi$ and known $\underline{\Delta A}, \overline{\Delta A} \in \mathbb{R}^{n\times n}$.*

In the system (17), the uncertainties are due to the initial state, the noises, the disturbances and also to the fact that the matrix $\Delta A(\rho)$ belongs into the interval $[\underline{\Delta A}, \overline{\Delta A}]$ for all $t \ge 0$. The interval $[\underline{\Delta A}, \overline{\Delta A}]$ is easy to compute for a given set Π and known $\Delta A(\rho)$.

The goal is to design an interval observer for the system (17). Before introduction of the interval observer equations note that for a matrix $L \in \mathbb{R}^{n\times p}$ the system (17) can be rewritten as follows:

$$\dot{x} = [A_0 - LC]x + \Delta A(\rho(t))x + L[y - v(t)] + d(t),$$

and according to Lemma 1 and Assumption 3 we have for all $\rho \in \Pi$:

$$\underline{\Delta A}^+\underline{x}^+ - \overline{\Delta A}^+\underline{x}^- - \underline{\Delta A}^-\overline{x}^+ + \overline{\Delta A}^-\overline{x}^- \le \Delta A(\rho)x \quad (18)$$
$$\le \overline{\Delta A}^+\overline{x}^+ - \underline{\Delta A}^+\overline{x}^- - \overline{\Delta A}^-\underline{x}^+ + \underline{\Delta A}^-\underline{x}^-$$

provided that $\underline{x} \le x \le \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$.

The following interval observer structure is proposed [46, 38] for the LPV system (17):

$$\begin{aligned}
\dot{\underline{x}} &= [A_0 - \underline{L}C]\underline{x} + [\underline{\Delta A}^+\underline{x}^+ - \overline{\Delta A}^+\underline{x}^- \\
&\quad -\underline{\Delta A}^-\overline{x}^+ + \overline{\Delta A}^-\overline{x}^-] + \underline{L}y - |\underline{L}|VE_p + \underline{d}(t), \\
\dot{\overline{x}} &= [A_0 - \overline{L}C]\overline{x} + [\overline{\Delta A}^+\overline{x}^+ - \underline{\Delta A}^+\overline{x}^- \quad (19)\\
&\quad -\overline{\Delta A}^-\underline{x}^+ + \underline{\Delta A}^-\underline{x}^-] + \overline{L}y + |\overline{L}|VE_p + \overline{d}(t), \\
&\quad \underline{x}(0) = \underline{x}_0,\ \overline{x}(0) = \overline{x}_0.
\end{aligned}$$

Note that due to the presence of $\underline{x}^+, \underline{x}^-, \overline{x}^+$ and $\overline{x}^-$, the interval observer (19) is a globally Lipschitz *nonlinear* system. In addition, in (19) the dynamics of $\underline{x}$ and $\overline{x}$ are interrelated.

Theorem 3. [38] *Let assumption 3 be satisfied and the matrices $A_0 - \underline{L}C$, $A_0 - \overline{L}C$ be Metzler. Then the relations $\underline{x}(t) \le x(t) \le \overline{x}(t)$ hold for all $t \ge t_0$. Furthermore, if there exist $P \in \mathbb{R}^{2n\times 2n}$, $P = P^T \succ 0$ and $\gamma > 0$ such that the following Riccati matrix inequality is verified*

$$G^TP + PG + 2\gamma^{-2}P^2 + \gamma^2\eta^2 I_{2n} + Z^TZ \prec 0, \quad (20)$$

where $\eta = 2n||\overline{\Delta A} - \underline{\Delta A}||_{max}$, $Z \in \mathbb{R}^{s\times 2n}$, $0 < s \le 2n$ and

$$G = \begin{bmatrix} A_0 - \underline{L}C + \underline{\Delta A}^+ & -\underline{\Delta A}^- \\ -\overline{\Delta A}^- & A_0 - \overline{L}C + \overline{\Delta A}^+ \end{bmatrix},$$

then $\underline{x}, \overline{x} \in \mathcal{L}_\infty^n$. In addition, the operator $[\underline{d}, \overline{d}] \to Z[\underline{x}, \overline{x}]$ in (19) has an L_2 gain less than γ.

Note that if there are no gains such that $A_0 - \underline{L}C$ and $A_0 - \overline{L}C$ are Metzler, then a transformation of coordinates T can be used for (19) in order to relax such requirement. The Riccati matrix inequality from Theorem 3 can be reformulated in terms of LMIs with respect to $\underline{L}$, $\overline{L}$ and P [38]. In addition, the observability property is not explicitly required in this section. However, without the detectability, the Riccati inequality (20) may not have a solution.

Example 1. Consider a nonlinear system [38]:

$$\dot{x} = \begin{bmatrix} \epsilon\cos t & 1 + \epsilon\sin x_3 & \varepsilon\sin x_2 \\ \varepsilon\sin x_3 & -0.5 + \epsilon\sin t & 1 + \epsilon\cos 2t \\ \varepsilon\sin x_2 & 0.3 + \varepsilon\cos 2t & -1 + \epsilon\sin t \end{bmatrix} x \quad (21)$$
$$+ \begin{bmatrix} 6\cos x_1 \\ \sin t + 0.1\sin x_3 \\ -\cos 3t + 0.1\sin 2x_2 \end{bmatrix},\ y = x_1 + v(t), \quad (22)$$

where $\epsilon = 0.01$ and $\varepsilon = 0.001$. We assume that $V = 0.1$, and for simulation we take $v(t) = V(\sin 5t + \cos 3t)/2$.

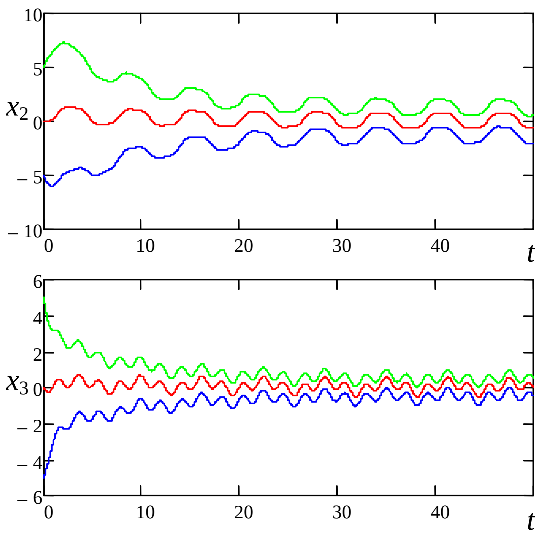

Fig. 1: The bounds (green and blue) of the unmeasured states x_2 and x_3 of the LPV system (21) - (22), the actual (unknown) state in red.

For initial conditions $|x_i(0)| \leq 5$ the system has bounded solutions. This system can be presented in the form of (17) for

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.5 & 1 \\ 0 & 0.3 & -1 \end{bmatrix}, \overline{\Delta A} = \begin{bmatrix} \epsilon & \epsilon & \varepsilon \\ \varepsilon & \epsilon & \epsilon \\ \varepsilon & \varepsilon & \epsilon \end{bmatrix} = -\underline{\Delta A},$$

$$\underline{b}(t,y) = \begin{bmatrix} 6\underline{f}(y) \\ \sin t - 0.1 \\ -\cos 3t - 0.1 \end{bmatrix}, \overline{b}(t,y) = \begin{bmatrix} 6\overline{f}(y) \\ \sin t + 0.1 \\ -\cos 3t + 0.1 \end{bmatrix},$$

$$\underline{f}(y) = \begin{cases} \cos y \cos V & \text{if } \cos y \geq 0 \\ \cos y & \text{if } \cos y < 0 \end{cases} - |\sin y| \sin V,$$

$$\overline{f}(y) = \begin{cases} \cos y & \text{if } \cos y \geq 0 \\ \cos y \cos V & \text{if } \cos y < 0 \end{cases} + |\sin y| \sin V$$

and a properly selected ρ, clearly assumption 3 is satisfied. A solution of the LMIs formulated for Theorem 3 gives the L_2 optimal solution [38]:

$$\underline{L} = \begin{bmatrix} 82.923 \\ -3e-4 \\ -4e-4 \end{bmatrix}, \overline{L} = \begin{bmatrix} 97.16 \\ -2e-5 \\ -1e-5 \end{bmatrix}$$

for $\gamma = 31.4$. The results of interval simulations for the variables x_2 and x_3 are given in Fig. 1 (the variable x_1 is omitted since it is available for measurements).

6 Case of LTV systems

LTV systems can be considered as a special case of LPV ones when the vector of scheduling parameters is a function of time and available at the design stage. Interval observers have been proposed for LTV systems in [47, 48], where different variants of transformations of time-varying systems to nonnegative forms are presented. Consider an LTV system described by:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + d(t), \\ y(t) = C(t)x(t) + v(t), \\ x(t_0) \in \mathbb{R}^n, \ t \geq t_0 \geq 0, \end{cases} \tag{23}$$

where $x(t) \in \mathbb{R}^n$, $d(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and $v(t) \in \mathbb{R}^p$ are respectively the state vector, an unknown but bounded input, the output vector and a bounded noise. Let $x(t_0) \in [\underline{x}_0, \overline{x}_0]$ for some known $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$; $d(t) \in [\underline{d}(t), \overline{d}(t)]$ for all $t \geq t_0$, where $\underline{d}, \overline{d} \in \mathcal{L}_\infty^n$; $v(t) \in [\underline{v}(t), \overline{v}(t)]$ for all $t \geq t_0$, where $\underline{v}, \overline{v} \in \mathcal{L}_\infty^p$.

In [48], an effective technique has been proposed to build an interval observer for systems described by (23).

Assumption 4. *There exist bounded matrix functions $L : \mathbb{R} \to \mathbb{R}^{n \times p}$, $M : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$, $M(\cdot) = M(\cdot)^T \succ 0$ such that for all $t \geq t_0$,*

$$\begin{aligned} &\dot{M}(t) + D(t)^T M(t) + M(t)D(t) \prec 0, \\ &D(t) = A(t) - L(t)C(t). \end{aligned}$$

Assumption 4 is a conventional requirement for LTV systems [49]. Under this assumption, the observer gain $L(t)$ and the matrix function $M(t)$ are such that the stability of the LTV system $\dot{x}(t) = D(t)x(t)$ can be proven by taking $V(t) = x(t)^T M(t)x(t)$ as a Lyapunov function. It determines the output stabilization conditions of the system (23) which can be rewritten as:

$$\begin{cases} \dot{x}(t) = D(t)x(t) + \tilde{d}(t), \\ y(t) = C(t)x(t) + v(t) \end{cases} \tag{24}$$

with $\tilde{d}(t) = d(t) - L(t)v(t) + L(t)y(t)$. When the gain $L(t)$ is computed such that the closed loop matrix $D(t)$ is stable and Metzler, the observer design is similar to the ones for LTI and LPV systems. In [48], such a restrictive condition has been avoided. The methodology is based on the D-similarities approach developed in [50, 51, 52], where time-varying matrices are transformed under a Metzler form.

Proposition 1. *There exists a time-varying transformation $z = T(t)x$ transforming $D(t)$ into a Metzler matrix $\Gamma(t)$:*

$$\Gamma(t) = T(t)\left(D(t)T^{-1}(t) - d(T^{-1}(t))/dt\right) \tag{25}$$

where $T(t) = P(t)L^{-1}(t)$ *with*

$$\Gamma(t) = \begin{bmatrix} \lambda_1(t) & 1 & \cdots & 0 \\ 0 & \lambda_2(t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_n(t) \end{bmatrix}. \quad (26)$$

The elements $\lambda_i(t)$ are called Essential D-eigenvalues of $D(t)$ (or ED-eigenvalues, see Definition 3.1 in [50]).

Theorem 4. [48] *Let assumption 4 be satisfied for (23) and* $\|y(t)\| \leq Y$ *for all* $t \geq t_0$ *for a known constant* $Y > 0$. *Given the matrix* T *defined in Proposition 1 and assume that* $\exists\ M_1 \in \mathbb{R}_+$ *such that* $\|T(t)\| + \|T^{-1}(t)\| \leq M_1$ *for all* $t \geq t_0$. *Then,*

$$\begin{aligned} \dot{\underline{z}}(t) &= \Gamma(t)\underline{z}(t) + \underline{d}_{obs}(t) + \underline{\Psi}_{obs}(t) + T_{obs}(t)y(t) \\ \dot{\overline{z}}(t) &= \Gamma(t)\overline{z}(t) + \overline{d}_{obs}(t) + \overline{\Psi}_{obs}(t) + T_{obs}(t)y(t) \end{aligned}$$

is an interval observer for (24) and

$$\underline{z}(t) \leq T(t)x(t) \leq \overline{z}(t),\ \forall t \geq t_0,$$

where $\underline{d}_{obs}(t) = T^+(t)\underline{d}(t) - T^-(t)\overline{d}(t)$, $T_{obs}(t) = T(t)L(t)$, $\underline{\Psi}_{obs}(t) = T^-_{obs}(t)\underline{v}(t) - T^+_{obs}(t)\overline{v}(t)$, $\overline{d}_{obs}(t) = T^+(t)\overline{d}(t) - T^-(t)\underline{d}(t)$, $\overline{\Psi}_{obs}(t) = T^-_{obs}(t)\overline{v}(t) - T^+_{obs}(t)\underline{v}(t)$.

The estimation in the original coordinates x is obtained using Lemma 1 with $\underline{z}$ and $\overline{z}$.

7 Case of switched systems

Consider a continuous-time switched system described by:

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + w(t) \\ y_m(t) = C_{\sigma(t)}x(t) + v(t) \end{cases}, \sigma(t) \in \mathcal{I} \quad (27)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y_m \in \mathbb{R}^p$, $w \in \mathbb{R}^n$, $v \in \mathbb{R}^p$ are the state vector, the input, the output, the disturbance and the measurement noise respectively. $q = \sigma(t)$ is the index of the active subsystem and is assumed to be known. A_q, B_q and C_q are constant matrices of appropriate dimensions.

The aim is to derive two signals $\underline{x}(t)$ and $\overline{x}(t)$ such that $\underline{x}(t) \leq x(t) \leq \overline{x}(t), \forall t \geq 0$ holds despite the disturbances and uncertainties provided that $\underline{x}_0 \leq x_0 \leq \overline{x}_0$ is satisfied. As in the previous sections, we assume that the measurement noise and the state disturbances are assumed to be unknown but bounded with *a priori* known bounds such that $-\overline{w}(t) \leq w(t) \leq \overline{w}(t)$, and $-VE_p \leq v(t) \leq VE_p, \forall t \geq 0$, where $\overline{w} : \mathbb{R}_+ \to \mathbb{R}^n$, $\overline{w}(t) \in \mathcal{L}^n_\infty$ and $V > 0$.

As in [53, 54], assume that there exist gains L_q such that the matrices $(A_q - L_qC_q)$ are Metzler for all $q \in \mathcal{I}$, where the matrices L_q $(q \in \mathcal{I})$ denote the observer gains associated with each subsystem q.

A candidate interval observer structure to compute $\overline{x}$ and $\underline{x}$ is described by:

$$\begin{cases} \dot{\overline{x}} = (A_q - L_qC_q)\overline{x} + B_qu + \overline{w} + L_qy_m + |L_q|VE_p \\ \dot{\underline{x}} = (A_q - L_qC_q)\underline{x} + B_qu - \overline{w} + L_qy_m - |L_q|VE_p \end{cases} \quad (28)$$

The following theorem gives the conditions for achieving the desired design goal.

Theorem 5. *[55] Given* $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$. *Let* Assumption 1 *be satisfied and* $x \in \mathcal{L}^n_\infty$. *If there exist* $\lambda, \eta \in \mathbb{R}^n$, $\lambda > 0$, $\eta > 0$, *matrices* $W_q \in \mathbb{R}^{n\times n}$ *and positive scalars* α_q, $\forall q \in \mathcal{I}$ *such that*

$$\begin{cases} A_q^TS + SA_q - C_q^TW_q^T - W_qC_q + \alpha_qS \prec 0 \\ A_q^TS - C_q^TW_q^T + diag(\eta) \geq 0 \\ S = diag(\lambda) \end{cases} \quad (29)$$

then, the solutions of (28) and (27) satisfy $\underline{x}, \overline{x} \in \mathcal{L}^n_\infty$ *and the inclusion*

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t)$$

provided that $\underline{x}_0 \leq x_0 \leq \overline{x}_0$, *where* $L_q = S^{-1}W_q$.

Remark 1. The system (28) is initialized with the initial conditions $\underline{x}_0$ and $\overline{x}_0$ for the first active subsystem. At the switching time instant t_{i+1}, the output of the previous active subsystem $(q = \sigma(t_i))$ is used to initialize (28) with the subsystem $(q = \sigma(t_{i+1}))$. ■

In the LMI (29), only the stability of the interval observer (28) is ensured. As in the case of non-switched systems, this LMI can be reformulated to find observer gains L_q minimizing the total interval estimation error $\overline{x} - \underline{x}$ as in [56, 27, 40]. The main limitation of the approach detailed in this section is the existence of gains L_q such that the matrices $(A_q - L_qC_q)$ are Metzler for all $q \in \mathcal{I}$. To overcome this problem, some techniques which consist in finding a change of coordinates that transforms the observation errors into cooperative forms are proposed in [55, 57, 58]. The changes of coordinates proposed for instance in [29, 59] for continuous systems can be used to transform the matrices $(A_q - L_qC_q)$ into a Metzler form.

Example 2. Consider a system (27) with $\mathcal{I} = \overline{1, 2}$ and

$$A_1 = \begin{bmatrix} -3 & 0 & 0.5 \\ 0 & -2.754 & 0 \\ 0 & 0 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -2.5 & 0 & 0.2 \\ 0 & -0.452 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T, C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$w(t)$ and $v(t)$ are uniformly distributed bounded signals such that $-\overline{w} \leq w(t) \leq \overline{w}$ with $\overline{w} = [0.015\ 0.015\ 0.015]^T$ and $-\overline{V}E_p \leq v(t) \leq \overline{V}E_p$ with $\overline{V} = 0.05$.
For this simple example, it is not easy to compute gains L_q satisfying (29) and the interval observer (28) cannot be applied. However, it is possible to determine matrices P_1, P_2 such that $P_1(A_1 - L_1C_1)P_1^{-1}$ and $P_2(A_2 - L_2C_2)P_2^{-1}$ are Metzler [58]. Using the Matlab LMI toolbox, one feasible solution is given by: $L_1 = \begin{bmatrix} 0.9226 & 0 & -0.0111 \end{bmatrix}^T$, $L_2 = \begin{bmatrix} 0 & 1.9007 & 0 \end{bmatrix}^T$. The results of simulation of the obtained observer are depicted in Fig. 2 where solid lines present the state and dashed lines present the estimated bounds. The input signal $u(t)$ is a square signal with frequency 2Hz and magnitude 0.5. The switching between the two subsystems is governed by the switching signal plotted in Fig. 3. The interval observer (28) is used with a change of coordinates. It is initialized with $-\overline{x}_0 \leq x_0 \leq \overline{x}_0$ with $\overline{x}_0 = \begin{bmatrix} 0.2 & 0.2 & 0.2 \end{bmatrix}^T$. The simulations show that the actual state belongs into the interval defined by the lower and upper bounds (inclusion property). In addition, the upper and lower observers are stable despite the switching instants.

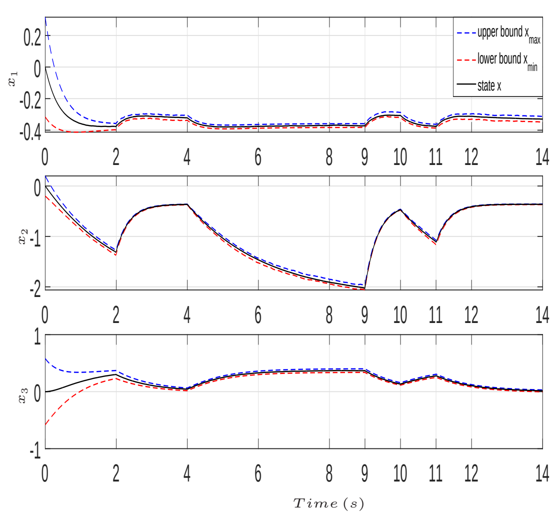

Fig. 2: Simulation results of the second example. The actual state and their bounds are plotted.

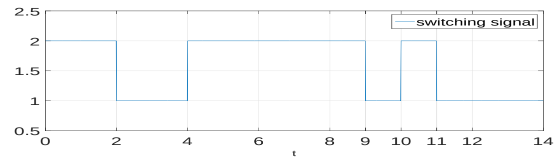

Fig. 3: Switching signal.

8 Conclusion

Some recent results on interval observers for continuous-time systems have been presented in this overview. Interval observers can be considered as conventional pointwise observers with a point estimate and an uncertainty quantification respectively given by the midpoint and the radius of each interval. This overview can be completed in a further paper by some results on discrete-time systems [23, 24, 25, 26, 27, 28], joint unknown and input estimation [60, 61] and applications of interval observers in the field of robust control [62, 26, 63, 64, 65], diagnosis [66, 67, 68, 69], prognosis [70] and automotive [71, 72].

In addition, sliding mode control and other estimation techniques are well known for their compensation of matched disturbances and finite-time convergence. Combination of sliding mode differentiators and interval observers is investigated in [22]. More generally, the combination of standard and interval observers can be considered as an appealing research issue for future works. Furthermore, a lot of open problems are still open in the field of interval observers design, namely for time-delay [41], hybrid [73, 58] and distributed [74] systems. The optimization of the width of interval estimation remains also an interesting issue in the field of linear parameter varying systems.

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